# P-Ideal Dichotomy, Iterating Reals and Universal Structures of Cardinality $\aleph_1$

Juris Steprāns

Workshop on Set Theory and Its Applications in Topology
Casa Matemática Oaxaca — September 12, 2016

It is well known that countably saturated models are universal for models of cardinality $\aleph_1$. One way of stating the saturation property for graphs is the following.

### Definition

*A saturated graph on $\omega_1$ is a function $G : [\omega_1]^2 \to 2$ such that if the function $G_\eta : \eta \to 2$ is defined by $G_\eta(\xi) = G(\{\eta, \xi\})$ then*

$$\{ G_\eta \restriction \alpha \mid \alpha \in \eta \in \omega_1 \} = 2^\alpha$$

*for each $\alpha \in \omega_1$.*

It follows that the existence of a saturated graph is equivalent to $2^{\aleph_0} = \aleph_1$. These raises the question of whether it is possible to have a universal graph of cardinality $\aleph_1$ in the absence of the Continuum Hypothesis.

This was answered positively by Shelah in 1984 in *On universal graphs without instances of CH*, Ann. Pure Appl. Logic (26), 75–87. But one might also ask which of the following weak forms of saturation yield universal graphs.

**Definition**

*Given ideals $\mathcal{I}_\alpha$ on $2^\alpha$, an $\mathcal{I}_\alpha$-saturated graph on $\omega_1$ is a function $G : [\omega_1]^2 \to 2$ such that $\{ G_\eta \restriction \alpha \mid \alpha \in \eta \in \omega_1 \} \notin \mathcal{I}_\alpha$ for each $\alpha \in \omega_1$.*

So one talk about meagre or null saturated graphs. For the complete graph on $\omega_1$ coloured in countably many colours there is now the notion of a bounded saturated colouring.

Let **PT** denote Miller's perfect set forcing.

### Definition

*If $G \subseteq \mathbf{PT}$ is generic over $V$ define $\mathcal{S}(\mathbf{PT})$ to be the set of all $S \in [\omega_1]^{\aleph_0}$ such that there is $T \in G$ and $\psi : T \to [\omega_1]^{<\aleph_0}$ such that*

1. $\psi \in V$
2. $\psi(s) \cap \psi(t) = \emptyset$ *unless* $s = t$
3. $T \Vdash_{\mathbf{PT}}$ "$\dot{S} = \bigcup_{j=0}^{\infty} \psi(r_{\dot{G}} \restriction j)$" *where* $r_G : \omega \to \omega$ *is the generic real obtained from the generic set* $G$.

### Lemma

*Let $G \subseteq \mathbf{PT}$ be generic over $V$ and $S \subseteq \omega_1$ in $V[G]$. Then the following are equivalent:*

1. $S \in \mathcal{S}(\mathbf{PT})$
2. *for all $F : \omega \times \omega \to \omega_1$ in $V$ such that the mapping $j \mapsto F(n, j)$ is one-to-one for all $n \in \omega$, there is $g : \omega \to \omega$ in $V$ such that for all $n \in \omega$ there is $k \leq g(n)$ such that $F(n, k) \notin S$*
3. *for all $F : \omega \times \omega \to \omega_1$ in $V$ such that the mapping $j \mapsto F(n, j)$ is one-to-one for all $n \in \omega$, there is a one-to-one function $g : \omega \times \omega \to \omega$ in $V$ such that $F(n, g(n, m)) \notin S$ for all $(n, m) \in \omega \times \omega$.*

*So $\mathcal{S}(\mathbf{PT})$ is an ideal containing no infinite set from $V$.*

### Lemma

*If $G$ is $\mathbf{PT}$ generic over $V$ then $\mathcal{S}(\mathbf{PT})$ is a P-ideal in $V[G]$.*

A fusion argument and the disjointness property of $\mathcal{S}(\mathbf{PT})$ it is possible to prove the following.

**Lemma**

*If $G$ is $\mathbf{PT}$ generic over $V$ and $S \in \mathcal{S}(\mathbf{PT})$ and $f : S \to 2$ is a function in $V[G]$ then there is $T \in G$ and $\psi$ defined on $T$ with disjoint range and $f^*$ such that*

1. $T \Vdash_{\mathbf{PT}}$ "$\dot{S} = S_\psi$"
2. $f^* : \bigcup_{t \in T} \psi(t) \to 2$
3. *if* $t \in T$ *then* $T[t] \Vdash_{\mathbf{PT}}$ "$\dot{f} \restriction \psi(t) = f^* \restriction \psi(t)$" *and, hence,* $T \Vdash_{\mathbf{PT}}$ "$\dot{f} = f^* \restriction S$".

## Lemma

*If $G$ is* **PT** *generic over $V$ and $S \in \mathcal{S}(\mathbf{PT})$, $S \subseteq \xi \in \omega_1$, $f : S \to 2$ is a function in $V[G]$ and $Z \subseteq 2^\xi$ is nowhere meagre, then there is $z \in Z$ such that $f \subseteq z$.*

**Proof:** Use the preceding lemma to get $T$ and $f^*$. Then the set of $h \in 2^\xi$ such that for all $k \in \omega$ and for all $t \in \mathbf{split}_k(T)$

$$\left|\left\{s \in \mathbf{split}_{k+1}(T) \mid s \supseteq t \text{ and } f^* \restriction \psi(s) \subseteq h\right\}\right| = \aleph_0$$

is a dense $G_\delta$ above the restriction of $f^*$ to the root of $T$ because of the disjointness property of $\psi$.

# Applying the P-ideal dichotomy

**Lemma**

*If $G$ is $\mathbf{PT}$ generic over $V$ then no uncountable subset of $\omega_1$ is orthogonal to $\mathcal{S}(\mathbf{PT})$ in $V[G]$; and hence, not even the union of countably many sets orthogonal to $\mathcal{S}(\mathbf{PT})$.*

**Proof:** Suppose that $Z$ is a $\mathbf{PT}$-name such that $T \Vdash_{\mathbf{PT}}$ "$Z \in [\omega_1]^{\aleph_1}$". It suffices to construct a sequence of conditions $T_n \in \mathbf{PT}$ and ordinals $\zeta_n$ such that:

- $T_0 = T$,
- $T_{n+1} \leq_n T_n$ for each $n$
- $T_n\langle u_j\rangle \Vdash_{\mathbf{PT}}$ "$\zeta_j \in Z$" for each $j \in n$
- the mapping $j \mapsto \zeta_j$ is one-to-one.

## Theorem (Abraham & Todorcevic)

*If $\mathcal{I}$ is a P-ideal on $\omega_1$ then there is a partial order $\mathbb{P}_{\mathcal{I}}$, that adds no reals, even when iterated with countable support, such that $\mathbb{P}_{\mathcal{I}}$ adds a set $Z \subseteq \omega_1$ such that for any $W \subseteq \omega_1$ which is not the union of countably many sets orthogonal to $\mathcal{I}$*

$$1 \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{"}\dot{Z} \cap W \neq \emptyset\text{"} \tag{1}$$

$$1 \Vdash_{\mathbb{P}_{\mathcal{I}}} \text{"}(\forall \eta \in \omega_1)\ \dot{Z} \cap \eta \in \mathcal{I}\text{"}. \tag{2}$$

**Proof:** To get (1) is implicit in the proof of Abraham and Todorcevic.

## Theorem

*Let $V$ be a model of set theory and suppose that $U : \omega_1^2 \to 2$ is a symmetric, category saturated function in $V$ and that $G \subseteq \mathbf{PT}$ is generic over $V$. In $V[G]$ let $H \subseteq \mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ be generic over $V[G]$. Then in $V[G][H]$ the function $U$ is universal.*

**Proof:** Using the previous theorem in $V[G]$ there is $R \subseteq \omega_1$ such that $[R]^{\aleph_0} \subseteq \mathcal{S}(\mathbf{PT})$ and $R \cap Y \neq \emptyset$ for each uncountable $Y \in V[G]$. Given $W : [\omega_1]^2 \to 2$, construct by induction embeddings $e_\eta : \eta \to R$ of $W \restriction \eta^2$ into $U$ such that $e_\eta \subseteq e_\zeta$ if $\eta \leq \zeta$.

Since limit stages of the induction are trivial, it suffices to show that given $e_\eta$ there is $e_{\eta+1}$ as required. Let $S$ be the range of $e_\eta$ and suppose that $S \subseteq \xi$. Then $S \in [R]^{\aleph_0} \subseteq \mathcal{S}(\mathbf{PT})$. Let $f : S \to 2$ be defined by $f(\sigma) = W(e_\eta^{-1}(\sigma), \eta)$ and note that $f \in V[G]$ since $V[G]$ and $V[G][H]$ have the same reals. Recall that **PT** preserves non-meagre sets.

It therefore follows that, recalling the notation from the first slide, $\{\gamma \in \omega_1 \mid f \subseteq U_\gamma\}$ is an uncountable set in $V[G]$. By the preceding theorem it is possible to find $\gamma \in R \setminus \xi$ such that $f \subseteq U_\gamma$ and, hence, $W(e_\eta^{-1}(\sigma), \eta) = f(\sigma) = U(\sigma, \gamma)$ for all $\sigma \in \mu$. Let $e_{\eta+1} = e_\eta \cup \{(\eta, \gamma)\}$.

### Corollary

*It is consistent with $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \aleph_2$ that there is a universal graph on $\omega_1$.*

The argument actually yields that if $V$ is any model of set theory in which there is a category saturated graph and if $G$ is **PT** generic over $V$ and $H$ is $\mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ generic over $V[G]$ then already in the model $V[G][H]$ the category saturated graph is universal. So if there were a model of $2^{\aleph_0} > \aleph_1$ with a category saturated graph such that forcing with $\mathbf{PT} * \mathbb{P}_{\mathcal{S}(\mathbf{PT})}$ does not collapse the continuum, then this would yield an even simpler method for obtaining a universal graph with the failure of the continuum hypothesis. But even the following question seems to be open.

**Question**

*Is there a model of set theory in which there is a non-meagre set of cardinality less than $2^{\aleph_0}$ such that forcing with* **PT** *over this model does not collapse the continuum?*

Applying similar arguments with Laver reals yields the following.

**Corollary**

*Consistently with $\mathfrak{b} = \mathfrak{d} = \aleph_2$ there is a universal graph on $\omega_1$.*

Applying similar arguments with certain $\omega^\omega$-bounding reals yields the following.

**Corollary**

*Consistently with $\mathfrak{b} = \mathfrak{d} = \aleph_1$ there is a universal graph on $\omega_1$.*