Spaces that are discretely generated at infinity

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  Discrete subspaces of maximal spaces are closed.

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Question

(Alas, Junqueria and Wilson, 2014) Is there a locally compact and discretely generated space with its one-point compactification NOT discretely generated?

First countable examples

Theorem

There is a first countable locally compact space with its one-point compactification not discretely generated if either:

(1) CH holds (Alas, Junqueira, Wilson, 2014) or

(2) there is a Souslin tree. (Aurichi 2009)

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How can we modify the CH example to obtain one under MA?

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- X is dg at $\omega \times {}^{\omega}2$ by first countability,
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- X is NOT discretely generated at $\{F\}$.

The harder part

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Notice that the $\mathfrak{p} = \mathrm{cof}(\mathcal{M})$ example exists under PFA.

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Case 1 A is countable.

Case 2 No countable subset of A has ∞ in its closure and ∞ has character ω_1 (in $X \cup \{\infty\}$).

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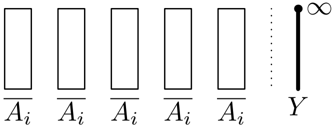

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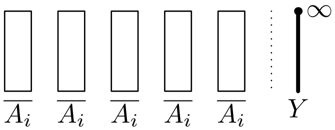

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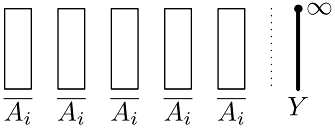

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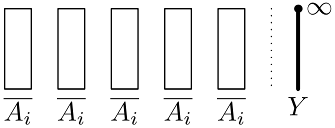

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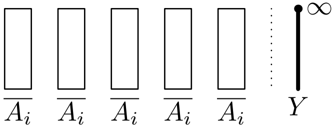

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Free ω_1-sequence

A sequence $\{x_\alpha : \alpha < \omega_1\} \subset K$ is a free ω_1-sequence if for every $\beta < \omega_1$,

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Lemma

Let K be a compact space and $p \in K$ such that $K \setminus \{p\}$ is countably tight, p is not isolated and p **is not in the closure of any countable discrete subset of** K. *Then there is a free ω_1-sequence in K such that p is its only complete accumulation point.*

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Then $\{a(n, f_\alpha(n)) : n \in E_\alpha\}$ converges to ∞.

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$q \leq p$ if $H_p \subset H_q$ and $\mathcal{N}_p \subset \mathcal{N}_q$

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Case 2: No countable subset of A has ∞ in its closure and ∞ has character ω_1.

$\mathbb{P}$ is proper (we will not prove this). Given a generic filter G,

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