Sequential groups, small and large

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Convergence

Definition: sequential spaces

A space X is called *sequential* if for every $A \subseteq X$ such that $\overline{A} \neq A$ there is a $C \subseteq A$ such that $C \to x \notin A$.

Trying to be more constructive

Definition: Sequential Closure

Let $A \subseteq X$. Define $[A]' = \{ x \in X : C \to x \text{ for some } C \subseteq A \}$. Put $[A]_\alpha = \cup\{[A_\beta]' : \beta < \alpha\}$ for $\alpha \leq \omega_1$.

Now X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subseteq X$.

Definition: Sequential Order

Define $\mathrm{so}(X) = \min\{ \alpha \leq \omega_1 : [A]_\alpha = \overline{A} \text{ for every } A \subseteq X \}$.

Sequential topological groups

Recall that X is *Fréchet* if $\mathrm{so}(X) \leq 1$. Can frist countable be weakened to Fréchet for separable groups in the well-known Birkhoff-Kakutani metrization theorem (Malykhin, 1978)?

Theorem [ZFC+ϵ]

Countable non first countable Fréchet groups exist consistently.

The celebrated answer to Malykhin's question.

Theorem (M. Hrušák and U.A. Ramos-García, 2012)

Every countable Fréchet group may be metrizable.

For general sequential groups, P. Nyikos asked

Question (P. Nyikos, 1980)

Is there a sequential group X such that $1 < \mathrm{so}(X) < \omega_1$?

Sequential groups X such that $\mathrm{so}(X) = \omega_1$ exist in ZFC.

As in the case of Malykhin's question, the answer is consistent.

Theorem [CH] (AS, 1998)

For any $\alpha \leq \omega_1$ there is a sequential group X such that $\mathrm{so}(X) = \alpha$.

Replacing countable with countably compact, Shakhmatov asked

Question (D. Shakhmatov, 1990)

Is there a countably compact sequential non Fréchet group?

All compact sequential (or countably tight) groups are first countable (= metrizable by Birkhoff-Kakutani). On the other hand

A countably compact Fréchet group

The $\sum$-power $\sum_{\omega_1} \mathbb{S}$ of the unit circle is countably compact, Fréchet, and not first countable.

Both questions have a consistent negative answer.

Theorem (AS, 2014)

It is consistent that there are no sequential non Fréchet groups whose sequential order is $< \omega_1$ or that are countably compact.

Note, that unlike the Fréchet case, there is no obvious reduction from the separable to the countable groups.

Natural questions

Does every sequential non Fréchet group have a nontrivial (non discrete, or non Fréchet) countable sequential subgroup? Does it have a quotient group of countable pseudocharacter that is not Fréchet?

Countable sequential groups are of independent interest.

Analytic topologies

For a countable X, the topology $\tau \subseteq 2^X$ can be viewed as a subset of the *Cantor cube* 2^ω. One can then consider the complexity of τ.

Definition: analytic spaces

A countable topological space X is called *analytic* if its topology is a continuous image of the irrationals, $\mathbb{N}^\omega$

Every *Borel* set is analytic. A useful source of examples is given by

Definition: topologies determined by families

Let X be a space and $\mathcal{P}$ be a family of its subspaces. The topology of X is said to be *determined* by $\mathcal{P}$ if $U \subseteq X$ is open if and only if $U \cap P$ is open in P for every $P \in \mathcal{P}$. If every element of $\mathcal{P}$ is compact, X is called a k-space. If, in addition, $\mathcal{P}$ is countable, X is said to be k_ω. All countable k_ω-spaces are analytic ($F_{\sigma\delta}$).

Analytic sequential groups

As it turns out, Malykhin's question has an *effective* resolution.

Theorem (S. Todorčević and C. Uzcátegui, 2001)

Every analytic Fréchet group is metrizable.

This naturally leads to an effective version of Niykos' question.

Question (S. Todorčević and C. Uzcátegui, 2001)

What are the possible sequential orders of analytic sequential groups?

A question about the size of the class of nice analytic spaces.

Question (S. Todorčević and C. Uzcátegui, 2001)

Do there exist uncountably many non homeomorphic analytic homogeneous spaces of sequential order ω_1?

k_ω-groups and scatteredness rank

The definition below is a key to the structure of k_ω-groups.

Definition: scatteredness rank

Put $[X]^- = X \setminus \{ x \in X \mid x \text{ is isolated in } X \}$, $[X]^0 = X$, $[X]^\alpha = \cap_{\beta<\alpha}[[X]^\beta]^-$. X is called *scattered* if $[X]^\alpha = \varnothing$ for some α. The smallest such α is the *scatteredness* of X. The scatteredness *rank* of X, $\mathrm{sc}(X)$ is the smallest α such that every compact subspace of X has the scatteredness of $\leq \alpha$. $\mathrm{sc}(X)$ is defined for every countable X and for such spaces $\mathrm{sc}(X) \leq \omega_1$.

together with the following folklore construction

Example: countable k_ω groups

Any countable non discrete topological group can be turned into a k_ω group of arbitrarily high scatteredness rank $< \omega_1$.

Zelenyuk's classification of k_ω-groups

The elegant result below classifies countable k_ω-groups.

Theorem (V. Zelenyuk, 1995)

Two countable k_ω groups $X \sim Y$ if and only if $\mathrm{sc}(X) = \mathrm{sc}(Y)$.

The next proposition is an easy corollary of a more general statement.

Theorem (AS, 1998)

If X is a countable k_ω group then $\mathrm{so}(X) \in \{0, \omega_1\}$.

Using the easy direction of Zelenyuk's theorem and the result above one obtains the answer to the second question of S. Todorčević and C. Uzcátegui.

Corollary

There are ω_1 analytic k-group topologies of sequential order ω_1.

Analytic sequential groups, continued

The next result provides a closer look at analytic group topologies.

Theorem (AS, 2016)

Every analytic sequential group is either first countable or k_ω.

This corollary answers another question of S. Todorčević and C. Uzcátegui.

Corollary

For a sequential analytic group X, $\mathrm{so}(X) \in \{0, 1, \omega_1\}$.

A. Dow (2014) constructed a countable Fréchet space without a countable π-base answering a question of I. Juhasz.

Theorem (AS, 2016)

Every Fréchet analytic space has a countable π-base.

Products of analytic sequential spaces

The non trivial direction of Zelenyuk's result together with a theorem above provide some additional insight into analytic groups.

Corollary

There are exactly ω_1 nonhomeomorphic analytic sequential group topologies. If X and Y are analytic sequential groups then $X \times Y \sim X$ or $X \times Y \sim Y$. In particular $X^n \sim X$ for every $n \in \omega$.

Parallels between topological groups and products are a well known phenomenon. Analytic spaces are no exception.

Theorem (AS, 2016)

The product $X \times S(\omega)$ with X sequential analytic is sequential if and only if X is a k_ω-space.

Subgroups of large sequential groups

It turns out k_ω-groups are a good starting point for the study of uncountable sequential groups as well.

Theorem (AS, 2016)

Let G be a sequential k_ω group and $G' \subseteq G$ be a subgroup of G such that $\overline{G'}$ is not Fréchet and $\overline{G'} \neq G'$. Then G' is not sequential.

The theorem above is a corollary of the following result which may be of independent interest.

Theorem (AS 2016)

Let G be a topological group and $G' \subseteq G$ be such that $\overline{G'}$ is a sequential non Fréchet k_ω group. Then G' contains a copy of $S(\omega)$ closed in G.

Adding a sequence to a k_ω group

Making some sequences converge. Recall that G is *boolean* of $a + a = 0$ for every $a \in G$.

Lemma: adding a sequence

Let G be a boolean k_ω group and $D \subseteq G$ be an infinite closed discrete subset of G. Let $a \in G$. Then there exists an infinite subset $C \subseteq D$ such that the finest group topology on G which is coarser than the original topology on G and such that $C \to a$ is a k_ω Hausdorff topology on G.

The conditions on G and D can be weakened to 'G is abelian and each mD is a closed discrete subset of G'. Applying this stronger version of the lemma one can construct real numbers $\langle r_n \mid n \in \omega \rangle$ such that $r_n \to \infty$ and a k_ω group topology on $\mathbb{R}$ coarser than the original topology and such that it is the finest group topology in which $r_n \to 0$.

This $\mathbb{R}$ becomes a sequential non Fréchet group (its sequential order is ω_1). It is also an easy observation that any countable closed subgroup of $\mathbb{R}$ must be cyclic and that every cyclic subgroup of $\mathbb{R}$ is dense in itself in the new topology.

Question

Does there exist a k_ω sequential group whose only sequential subgoups are discrete? In particular, can one pick r_n's above so that in the resulting topology $\mathbb{R}$ has no infinite closed cyclic subgroups?

It is known that such r_n cannot be all integers (D. Dikranjan and others).

An uncountable sequential group

Example: a 'nonreflecting sequential group' [$\diamondsuit$] (AS, 2016)

A sequential group G such that every countable sequential subgroup of G is discrete and every quotient of G is either Fréchet or has an uncountable pseudocharacter.

A G as above can be costructed to have the additional property that $G \times G$ is sequential and the only sequential subgroups of G are closed and uncountable. In addition, one can arrange for $\mathrm{so}(G) = \omega + 1$.

Question

Does there exist a sequential group G whose only countable sequential subgroups are finite and all of whose quotients of countable pseudocharacter are first-countable? All of whose quotients of pseudocharacter ω_1 are first-countable?

Open questions

Question

Can all countable sequential groups be analytic?

The next question was asked by M. Hrušak and U.A. Ramos-García

Question

Is it possible that some countable topologizable group admits a non-metrizable Fréchet group topology while another does not?

Its counterpart for sequential groups is

Question

Is it possible that only some topologizable groups admit a sequential topology with intermediate sequential order?

Open questions, continued

The folowing questions are specific to sequential groups.

Question

Is it consistent with ZFC that groups of intermediate sequential order α exist for some $\alpha \in \omega_1$ but not for all of them? Only finite α? Only infinite ones?

The effect of the *size* of the group is yet unclear.

Question

Is it consistent wih ZFC that there is an uncountable group of intermediate sequential order but there is no countable such?

Question

If G is a dense countable analytic subgroup of a sequential group H, is H a k_ω-group?