On hyperfiniteness of boundary actions of hyperbolic groups

Marcin Sabok

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This is joint work (in progress) with Jingyin Huang and Forte Shinko.
Definition ($\delta$-hyperbolic space)

Suppose $X$ is a geodesic metric space, $\delta > 0$ and $x, y, z \in X$. A geodesic triangle whose sides are geodesic segments $[x, y]$, $[y, z]$ and $[z, x]$ is called $\delta$-slim if any of the three above geodesic segments is in the $\delta$-neighborhood of the two remaining sides.
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Example

Note that if $X$ is a tree, then it is $\delta$-hyperbolic for any $\delta > 0$ as the geodesic triangles all look like tripods.
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In general, the smaller $\delta$ is, the more $\delta$-hyperbolic spaces “look like” trees.
Definition (hyperbolic group)

Suppose $\Gamma$ is a finitely generated group. $\Gamma$ is \textit{hyperbolic} if the Cayley graph of $\Gamma$ is $\delta$-hyperbolic for some $\delta > 0$. 

In the above definition, the Cayley graph is taken with respect to a given finite set of generators of $\Gamma$ and the metric on the graph is the graph metric. One can show that hyperbolicity does not depend on the choice of the generating set.

Examples

There are many examples of hyperbolic groups. The free groups $F_n$ are of course hyperbolic. All fundamental groups $\pi_1(M)$ of compact hyperbolic manifolds $M$ are hyperbolic.
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**Definition**

Given a hyperbolic space $X$ with a distinguished point $O$ we identify two geodesic rays $\gamma_1$ and $\gamma_2$ (write $\gamma_1 \sim \gamma_2$) if there exists a constant $K > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) < K$$

for all $t$. The **boundary of $X$**, denoted $\partial X$ is the set of all $\sim$-classes of geodesic rays in $X$. 
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Thus defined, $\partial X$ is just a set and it carries a natural compact topology.
Definition (Gromov product)

Given three points $x, y, z$ in a hyperbolic space $X$ we define the Gromov product as follows

$$(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$$
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Topology on the boundary

Given $p \in \partial X$ and $r > 0$ we define the neighborhood of $p$ as

$$\{ q \in \partial X : \exists \gamma \in q, \exists \gamma' \in p \quad \inf_{s,t \to \infty} (\gamma(s), \gamma'(t))_0 \geq r \}$$
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With the above topology, the boundary is a compact topological metrizable space.
If $\Gamma$ is a hyperbolic group, then $\partial \Gamma$ is the boundary of the Cayley graph of $\Gamma$ with $O$ being the neutral element $e$. 
Boundary of a hyperbolic group

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Suppose $\Gamma$ is a hyperbolic group and $p \in \partial \Gamma$. Let $\gamma \in p$ be a geodesic ray. For any $g \in \Gamma$ there exists a unique geodesic ray starting at $e$ which hits the geodesic $\gamma'(t) = g \cdot \gamma(t)$. Denote this geodesic ray by $g\gamma$. 
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Boundary action

The above $(g, p) \mapsto [g \gamma]_\sim$ induces an action of $\Gamma$ by homeomorphism on the boundary $\partial \Gamma$ which is called the boundary action.

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Definition

The *tail equivalence relation* $E_t$ is the equivalence relation defined at $2^\mathbb{N}$ as follows:

$$x E_t y \text{ if } \exists n, m \forall k \ x(n + k) = y(m + k)$$

Remark

It is not difficult to see that the tail equivalence relation is Borel-bireducible with the action of the free group $F_2$ on its boundary Cantor set.

Theorem (Dougherty–Jackson–Kechris)

The tail equivalence relation is hyperfinite.

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Question

Is the boundary action of every hyperbolic group hyperfinite?
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We can provide positive answer for a large class of hyperbolic groups.
Suppose $X$ is a geodesic metric space. Given three points $x, y, z \in X$ and geodesic segments $[x, y], [y, z], [z, x]$ consider a corresponding triangle $x', y', z'$ on the Euclidean plane with the lengths of $[x', y'], [y', z'], [z', x']$ equal to the corresponding lengths $[x, y], [y, z], [z, x]$. For any two points $p, q \in [x, y] \cup [y, z] \cup [z, x]$ there exist unique $p', q' \in [x, y] \cup [y, z] \cup [z, x]$ which divide the sides in the same proportion.
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Model triangles

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Definition

The space $X$ is CAT(0) if the for any $x, y, z, p, q \in X$ as above we have $d(p, q) \leq d_e(p', q')$ where $d_e$ is the Euclidean distance.
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Suppose a group acts on a cube complex. The action is *proper* if the stabilizers of all points are finite.
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Definition

The action of a group on a cube complex is *cocompact* if there are finitely many orbits.
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By a *cube complex* we mean a complex built of cubes $[0, 1]^n$. 
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Hyperbolic groups
For example, such a group is hyperbolic if and only if the complex is hyperbolic.
It turns out that many hyperbolic groups act properly and cocompactly on CAT(0) cube complexes.
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**Theorem (Bergeron–Wise, Kahn–Markovic)**

All fundamental groups of hyperbolic closed 3-manifolds admit proper cocompact actions on CAT(0) cube complexes.
Theorem (Huang–S.–Shinko)

If a hyperbolic group $\Gamma$ acts properly and cocompactly on a CAT(0) cube complex, then the boundary action of $\Gamma$ on $\partial \Gamma$ is hyperfinite.
Boundary of a complex

Given a proper and cocompact action of a hyperbolic group $\Gamma$ on a complex $X$ one can define the boundary of this action in a similar way as the boundary of the group.
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Hyperfiniteness

If a hyperbolic group acts properly and cocompactly on a complex, then the this induces an action on the boundary of the complex, which is hyperfinite if and only if the boundary action of the group is hyperfinite.
Theorem (Huang–S.–Shinko)

If a hyperbolic group $\Gamma$ acts properly and cocompactly on a CAT(0) cube complex $X$, then the induced action $\Gamma \curvearrowright \partial X$ is hyperfinite.