Diamonds are a Set Theorist's best friend

Víctor Torres-Pérez

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Set Theory and its Applications in Topology
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Therefore, $\mathrm{CH} \not\to \diamondsuit$.

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For example, $2^{\omega_1} = \omega_2$ implies $\diamondsuit_{\omega_2}(E^{\omega_2}_\omega)$.

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is stationary. The principle $\diamondsuit_{[\lambda]^\mu}$ states that there is a $\diamondsuit_{[\lambda]^\mu}$-sequence.

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$$\lambda^{\omega_1} = \begin{cases} \lambda & \textit{if } \operatorname{cof}\ \lambda > \omega_1, \\ \lambda^+ & \textit{if } \operatorname{cof}\ \lambda \leq \omega_1. \end{cases}$$

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An invariant (A, B, R) is *Borel*

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Case 1: $Y_{\alpha+1} = Y_\alpha \backslash A_\alpha$. Then $\omega \backslash A_\alpha \supseteq Y_\alpha \backslash A_\alpha = Y_{\alpha+1}$, and so $\omega \backslash A_\alpha \in \mathscr{U}_{\mathcal{Y}}$.

Case 2: $Y_{\alpha+1} = Y_\alpha \cap A_\alpha$. Then $A_\alpha \supseteq Y_\alpha \cap A_\alpha = Y_{\alpha+1}$, and so $A_\alpha \in \mathscr{U}_{\mathcal{Y}}$.

CH implies the Builder has a winning strategy in G_t

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Suppose otherwise, and pick $X \in [\omega]^\omega$ such that $X \subseteq^* Y_\alpha$ for every $\alpha < \omega_1$.

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We show that the sequence $\langle Y_\alpha : \alpha \in \omega_1 \rangle$ is a tower.
Suppose otherwise, and pick $X \in [\omega]^\omega$ such that $X \subseteq^* Y_\alpha$ for every $\alpha < \omega_1$. Let X_0, X_1 be two infinite disjoint subsets of X such that $X = X_0 \cup X_1$.

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Take $i \in \{0, 1\}$ such that $X_i \in \mathscr{U}_\mathcal{Y}$,

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Suppose otherwise, and pick $X \in [\omega]^\omega$ such that $X \subseteq^* Y_\alpha$ for every $\alpha < \omega_1$. Let X_0, X_1 be two infinite disjoint subsets of X such that $X = X_0 \cup X_1$. As we have mentioned, the filter generated $\mathscr{U}_{\mathcal{Y}}$ by $\langle Y_\alpha : \alpha < \omega_1 \rangle$ is an ultrafilter.
Take $i \in \{0, 1\}$ such that $X_i \in \mathscr{U}_{\mathcal{Y}}$, and let $\xi \in \omega_1$ such that $Y_\xi \subseteq^* X_i$. Then, $Y_\xi \cap X_{1-i}$ is finite, and so $X \not\subseteq^* Y_\xi$.

□

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Proof.

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Proof.

Given an infinite $\subseteq^*$-decreasing sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ with $\delta(s)$ limit,

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Proof.

Given an infinite $\subseteq^*$-decreasing sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ with $\delta(s)$ limit, we will define a strictly increasing sequence $\{l^s_i : i \in \omega\}$.

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Proof.

Given an infinite $\subseteq^*$-decreasing sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ with $\delta(s)$ limit, we will define a strictly increasing sequence $\{I^s_i : i \in \omega\}$. Fix an increasing sequence $\{\delta_i : i \in \omega\} \subseteq \delta(s)$ converging to $\delta(s)$.

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$$l^s_0 = \min\left(Y^s_{\delta_i}\right),$$

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$$l^s_0 = \min\left(Y^s_{\delta_i}\right),$$

and

$$l^s_{i+1} = \min\left(\bigcap_{j \leq i+1} Y^s_{\delta_j} \setminus (l^s_i + 1)\right).$$

$\diamondsuit(2, \neq)$ implies the Builder has a winning strategy in $G_{\mathfrak{t}}$

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$\diamondsuit(2, \neq)$ implies the Builder has a winning strategy in $G_{\mathfrak{t}}$

For a decreasing $\subseteq^*$-sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ of length an infinite limit ordinal and $C \subseteq \omega$ infinite,

$\diamondsuit(2,\neq)$ implies the Builder has a winning strategy in $G_{\mathfrak{t}}$

For a decreasing $\subseteq^*$-sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ of length an infinite limit ordinal and $C \subseteq \omega$ infinite, define $F(s, C)$ as follows:

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For a decreasing $\subseteq^*$-sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ of length an infinite limit ordinal and $C \subseteq \omega$ infinite, define $F(s, C)$ as follows:

$$F(s, C) = \begin{cases} 0 & \text{if } C \subseteq^* \{I^s_{2i} : i \in \omega\}, \\ 1 & \text{otherwise.} \end{cases}$$

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Let $g : \omega_1 \to 2$ be a $\diamondsuit(2, \neq)$-sequence for F.

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Suppose $s = \{Y^s_\xi : \xi < \delta(s)\}$ is a partial match with $\delta(s)$ an infinite limit ordinal.

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For a decreasing $\subseteq^*$-sequence $s = \{Y_\xi^s : \xi < \delta(s)\}$ of length an infinite limit ordinal and $C \subseteq \omega$ infinite, define $F(s, C)$ as follows:

$$F(s, C) = \begin{cases} 0 & \text{if } C \subseteq^* \{I_{2i}^s : i \in \omega\}, \\ 1 & \text{otherwise.} \end{cases}$$

Let $g : \omega_1 \to 2$ be a $\diamondsuit(2, \neq)$-sequence for F. We are going to use g to define a winning strategy for the Builder.
Suppose $s = \{Y_\xi^s : \xi < \delta(s)\}$ is a partial match with $\delta(s)$ an infinite limit ordinal. The Builder is going to choose $Y_{\delta(s)}$ as follows:

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Suppose $s = \{Y^s_\xi : \xi < \delta(s)\}$ is a partial match with $\delta(s)$ an infinite limit ordinal. The Builder is going to choose $Y_{\delta(s)}$ as follows:

$$Y_{\delta(s)} = \begin{cases} \{I^s_{2i} : i \in \omega\} & \text{if } g(\delta(s)) = 0, \\ \{I^s_{2i+1} : i \in \omega\} & \text{otherwise.} \end{cases}$$

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$\diamondsuit(2, \neq)$ implies the Builder has a winning strategy in G_{t}.

Let $s = \{Y_\xi^s : \xi < \omega_1\}$ be a complete match played by the Builder according to the strategy described above.

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Let $C \subseteq \omega$.

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Let $s = \{Y^s_\xi : \xi < \omega_1\}$ be a complete match played by the Builder according to the strategy described above.
Let $C \subseteq \omega$. Then if δ is an infinite limit ordinal such that $F(s{\restriction}_\delta, C) \neq g(\delta)$,

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Let $s = \{Y_\xi^s : \xi < \omega_1\}$ be a complete match played by the Builder according to the strategy described above.
Let $C \subseteq \omega$. Then if δ is an infinite limit ordinal such that $F(s\restriction_\delta, C) \neq g(\delta)$, it is straightforward to see that $C \not\subseteq^* Y_\delta$.

$\square$

The Builder having a winning strategy in G_t does not imply CH

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Theorem (Moore-Hrušák-Džamonja)

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CH *does not imply* $\diamondsuit_{\mathfrak{t}}$.

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Corollary

The Builder having a winning strategy in $G_{\mathfrak{t}}$ does not imply CH

We have the following:

Theorem (Moore-Hrušák-Džamonja)

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Corollary

$\diamondsuit(2,=) \not\leftarrow$ *the Builder has a winning strategy in the tower game* $G_{\mathfrak{t}}$.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

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Proof.

Assume CH.

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Proof.

Assume CH. Let $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ be a tower.

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Lemma

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Proof.

Assume CH. Let $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ be a tower. Let $(f_\alpha : \alpha < \omega_1)$ list all partial functions from $\omega \to \omega$ with infinite range.

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Proof.

Assume CH. Let $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ be a tower. Let $(f_\alpha : \alpha < \omega_1)$ list all partial functions from $\omega \to \omega$ with infinite range. Construct $(A_\alpha : \alpha < \omega_1)$ and $(B_\alpha : \alpha < \omega_1)$ such that for all α,

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- $A_\alpha \subseteq^* B_\alpha$, $B_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$,

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- $A_\alpha \subseteq^* B_\alpha$, $B_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$,
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- $A_\alpha \subseteq^* B_\alpha$, $B_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$,
- B_α is chosen according to a given rule, and
- if $\mathrm{ran}(f_\alpha \restriction_{B_\alpha})$ is infinite,

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- $A_\alpha \subseteq^* B_\alpha$, $B_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$,
- B_α is chosen according to a given rule, and
- if $\mathrm{ran}(f_\alpha \restriction_{B_\alpha})$ is infinite, then $\mathrm{ran}(f_\alpha \restriction_{A_\alpha})$ is almost disjoint from some Y_{β_α}.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

To choose A_α note that there is $\beta < \omega_1$ such that $\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha}$ is infinite because $\mathcal{Y}$ is a tower.

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To choose A_α note that there is $\beta < \omega_1$ such that $\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha}$ is infinite because $\mathcal{Y}$ is a tower. Now let $A_\alpha = f_\alpha^{-1}(\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha})$.

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$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

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$A_\alpha = f_\alpha^{-1}(\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha})$. This is as required.
Let $\mathcal{F}$ be the filter generated by the A_α.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_\mathfrak{t}$

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$A_\alpha = f_\alpha^{-1}(\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha})$. This is as required.
Let $\mathcal{F}$ be the filter generated by the A_α. Consider Laver forcing $\mathbb{L}_\mathcal{F}$ with $\mathcal{F}$.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_\mathfrak{t}$

To choose A_α note that there is $\beta < \omega_1$ such that $\mathrm{ran}(f_\alpha{\restriction}_{B_\alpha}) \setminus Y_{\beta_\alpha}$ is infinite because $\mathcal{Y}$ is a tower. Now let $A_\alpha = f_\alpha^{-1}(\mathrm{ran}(f_\alpha{\restriction}_{B_\alpha}) \setminus Y_{\beta_\alpha})$. This is as required.
Let $\mathcal{F}$ be the filter generated by the A_α. Consider Laver forcing $\mathbb{L}_\mathcal{F}$ with $\mathcal{F}$.
Assume the following:

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

To choose A_α note that there is $\beta < \omega_1$ such that $\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha}$ is infinite because $\mathcal{Y}$ is a tower. Now let $A_\alpha = f_\alpha^{-1}(\mathrm{ran}(f_\alpha \restriction_{B_\alpha}) \setminus Y_{\beta_\alpha})$. This is as required.
Let $\mathcal{F}$ be the filter generated by the A_α. Consider Laver forcing $\mathbb{L}_\mathcal{F}$ with $\mathcal{F}$.
Assume the following:

Claim

$\mathbb{L}_\mathcal{F}$ preserves $\mathcal{Y}$.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_{\mathfrak{t}}$.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_{\mathfrak{t}}$.

Proof.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_{\mathfrak{t}}$.

Proof.

Assume $\diamondsuit\left(E^{\omega_2}_{\omega_1}\right)$ and CH.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_{\mathfrak{t}}$.

Proof.

Assume $\diamondsuit(E^{\omega_2}_{\omega_1})$ and CH. Fix a tower $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ as above.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_{\mathfrak{t}}$.

Proof.

Assume $\diamondsuit(E^{\omega_2}_{\omega_1})$ and CH. Fix a tower $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ as above. Use the diamond to guess (initial segments of) names of strategies for the Builder.

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_\mathfrak{t}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_\mathfrak{t}$.

Proof.

Assume $\diamondsuit(E^{\omega_2}_{\omega_1})$ and CH. Fix a tower $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ as above. Use the diamond to guess (initial segments of) names of strategies for the Builder. Construct a finite support iteration

$\mathfrak{t} = \omega_1$ does not imply the Builder has a winning strategy in $G_{\mathfrak{t}}$

Corollary

It is consistent that $\mathfrak{t} = \omega_1$ and the Builder has no winning strategy in $G_{\mathfrak{t}}$.

Proof.

Assume $\diamondsuit\left(E^{\omega_2}_{\omega_1}\right)$ and CH. Fix a tower $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ as above. Use the diamond to guess (initial segments of) names of strategies for the Builder. Construct a finite support iteration $\left(\mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \omega_2\right)$.

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Open question:

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Open question:
The Builder has a winning strategy in the almost disjoint game $G_{\mathfrak{a}}$ $\not\leftarrow \mathfrak{a} = \omega_1$?

Thank you!