16w5072: Algorithmic Randomness Interacts with Analysis and Ergodic Theory

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1 Overview of the Field

This workshop brought together four logical approaches to understanding the algorithmic and computational content of mathematical theorems. In *Computable* or *Effective Analysis*, one aims to understand the extent to which classical mathematical theorems hold when the data and functions involved are required to be computable. In *Algorithmic Randomness*, one studies notions of randomness whereby "random" is understood as "unpredictable to a computational agent," in various senses. In *Reverse Mathematics*, one aims to understand the extent to which theorems can be proved in weak subsystems of second-order arithmetic, for which the set existence principles hold in restricted models, and for which the computational results fall into restricted complexity classes. Finally, in *Proof Mining*, one aims to uncover computational or quantitative information that is hidden or implicit in nonconstructive mathematical proofs.

Ergodic theory and topological dynamics have been considered from all four perspectives. These fields, closely related to analysis, provide a theory of iteration of operators. In the ergodic setting, the operator, defined on a probability space, is measure preserving; an example is rotations of the cycle by a fixed angle. In the topological setting, the space is compact and the operator continuous. It is then natural to consider what happens when the operators and spaces they act on are computable, in a suitable sense; and, in the measure-theoretic setting, what generic properties hold for algorithmically random elements. A goal and central outcome of this workshop was to bring together researchers in all these areas enumerated above, and explore the extent to which methods might be transferred between the different approaches.

2 Recent Developments and Open Problems

The area of algorithmic randomness has recently received a strong impetus from methods and concepts that originated in analysis or ergodic theory. For example, the analytic notion of density was used to answer a long-standing question about the computational power of "anti-random" sequences of bits (see Bienvenu et al. [7] for a summary); differentiability helped to show that a notion of randomness for a real number does not depend on the base it is represented in (Figueira and Nies [14]).

In the converse direction, algorithmic randomness can clarify the effective content of theorems from analysis or ergodic theory, in particular of theorems that state that a function or operator is well-behaved almost everywhere in the sense of measure. An example is Lebesgue's theorem that a nondecreasing function is differentiable at almost every point. Brattka, Miller and Nies [9] showed that an algorithmic version of this theorem corresponds to the notion of computable randomness. Computable versions of ergodic theorems, such as Birkhoff's, have been studied in a similar way by V'yugin [24], Franklin and Towsner [15], and many others. Schnorr randomness [23] is an important randomness notion that is weaker than computable randomness. Any Schnorr random point in an effective ergodic measure-preserving system is typical in the ergodic-theoretic sense. Another result due to Avigad [2] states that if x is a Schnorr random real and (a_i) is a computable sequence of distinct integers, the sequence $(a_i x)$ is equidistributed modulo 1, providing an algorithmic version of a well-known theorem of Weyl (Mathematische Annalen, 1916).

It turns out that algorithmic randomness also interacts strongly with reverse mathematics, the area of logic concerned with the strength of axioms needed to prove particular theorems. Algorithmic randomness notions give rise to interesting principles, and their strength can be studied through the corresponding analytic theorems.

The reverse mathematics of basic theorems of analysis, such as the mean value theorem, was investigated from the early 1980s. Slightly later, Simpson and Yu [25] examined the proof-theoretic strength of theorems from measure theory in the framework of reverse mathematics. Recent investigations of Greenberg, Nies, Yokoyama and others have helped determine the strength of Jordan's decomposition theorem, which states that every function of bounded variation is the difference of two non-decreasing functions. To date, though, not much is known about the role of randomness in reverse mathematics; the few existing results indicate there is much to be found.

Ergodic theory and topological dynamics have been successfully connected with computability and reverse mathematics: for instance, in the work of Blass, Hirst and Simpson (*Logic and Combinatorics*, 1987) and, more recently, work of Beiglböck and Towsner centered on Hindman's theorem [6]. There are several recent applications of computability to shifts in symbolic dynamics, such as in a paper by Hochman and Meyerovitch [16] in the Annals of Mathematics studying the entropy of multidimensional shifts of finite type, as well as work by Jeandel, Vanier and others relating subshifts to enumeration degrees.

Here are some of the topics that were discussed at the meeting.

2.1 Effective ergodic theory

One form of Birkhoff's ergodic theorem says that for a measure preserving operator T and a real valued integrable function f on an appropriate probability space (for instance, Cantor space $2^{\mathbb{N}}$), for almost every x the averages of the values $f(T^i(x))$ for $i = 0 \dots n$ converge as $n \to \infty$. Given an effective setting of this theorem, how much randomness of x is necessary to ensure this convergence? Combining results of V'yugin [24] and Franklin and Towsner [15] settles an important case, where T is computable and f is also computable: the randomness notion describing convergence is Martin-Löf's. This leaves open the case that f is merely lower semi-computable. Recently Miyabe, Nies and Zhang [19] showed that a randomness notion slightly stronger than Martin-Löf's is sufficient in this case. An intermediate case due to Hoyrup is given by a "nearly ergodic" computable operator T, which decomposes into a finite number of ergodic operators. Hoyrup has shown that these components are not necessarily computable.

Adam Day in 2014 has given a new proof, using descriptive string (or Kolmogorov) complexity, of a theorem due to Kolmogorov and Sinai independently: two isomorphic ergodic systems have the same entropy. This is only one example of new interactions between Kolmogorov complexity and ergodic theory. They were further explored at the meeting.

A considerable part of effective analysis is based on feasible computability, which means that the relevant objects are realistically computable, usually with polynomial time bounds. Nies [22], and independently Kawamura and Miyabe, obtained the analog of the above-mentioned Brattka-Miller-Nies result in the feasible setting: a real z is polynomial time random if and only if each polynomial time computable function is differentiable at z. Feasible ergodic theory, on the other hand, seems to be untouched. The meeting led to new perspectives here.

2.2 Algorithmic randomness and effective rates of convergence

One topic of the workshop was the extent to which convergence theorems in analysis have effective rates of convergence. For a computable sequence of measurable functions, a suitably effective rate of convergence implies that these functions converge pointwise on algorithmically random inputs. This provides a further link between algorithmic randomness and computable analysis, and brings the study of effective rates of convergence in ergodic theorems, martingale convergence theorems, and so on to bear on questions in algorithmic randomness. See for instance work of Avigad and Rute [4], and Kohlenbach and Leuştean [17].

2.3 Random closed sets and energy randomness

Barmpalias et al. [5] studied the following question: which reals can be points of random closed sets? Diamondstone and Kjos-Hanssen [12] related the question to Galton–Watson processes. This led them to effectivize a notion from probability, which itself was appropriated from mathematical physics. Call a real *energy s-random* if it is random for some measure μ with finite Riesz s-energy. Diamondstone and Kjos-Hanssen showed that energy s-randomness for the appropriate dimension s implies being a point of a random close set; they conjectured equivalence. Rute proved their conjecture, and also showed that the same notion (with a different dimension parameter) characterizes the zeros of Martin-Löf Brownian motions. This completed work started by Allen, Bienvenu and Slaman [1]. In a recent development, Miller and Rute (arXiv:1509.00524) have given a natural characterization of energy s-randomness in terms of Levin's a priori complexity. Similar to other directions pursued at the workshop, this shows a strong connection between notions imported from classical mathematics and those native to algorithmic randomness. To advance such interplay between different research specialties, it was helpful to bring the involved researchers (such as Miller and Rute) together for 5 days.

2.4 Reverse mathematics and randomness

Let RCA denote the usual "base theory", given by the axiom system stating some basic properties of the natural numbers, a certain amount of induction, and, essentially, the existence of all computable sets. Let $\mathcal{C}$ denote a randomness notion. Workshop participants such as Nies and Yokoyama studied the strength of the system RCA together with the statement that for each oracle X there is a bit sequence Y that is random relative to X in the sense of $\mathcal{C}$. For ML-randomness, the resulting principle is very weak Königs Lemma WWKL. For the stronger notion of 2-randomness, the corresponding principle has been studied by Avigad, Dean and Rute [3], Conidis and Slaman [10] as well as by Csima and Mileti [11] and more recently by Nies and Shafer.

Interestingly, the systems corresponding to different randomness notions can be equivalent. For instance, this is the case for computable randomness versus Schnorr randomness. On the other hand, analytic notions such as differentiability can be used to separate such axioms. We considered in particular notions slightly stronger than Martin-Löf randomness. The system for difference randomness (that is, Martin-Löf random not computing the halting problem) is equivalent to weak weak König's lemma WWKL, while the system for weak 2-randomness strictly implies WWKL. It would be interesting to determine the dividing line.

Understanding the effective content of theorems on topological dynamics can also determine the strength of the classical theorems in reverse mathematics. For instance, Adam Day has used this idea to show that another theorem due to Birkhoff, on the existence of almost periodic points, is strictly in between a system called arithmetic comprehension and weak König's lemma WKL. This theorem was proved in the special setting of a continuous operator on Cantor space – the strength of Birkhoff's result for continuous operators on other compact spaces remains open.

2.5 Density randomness and computational complexity

Several classical theorems assert that almost all reals share some property. In the overview section, we discussed differentiability. Another example is Lebesgue's density theorem. For any measurable set $P \subseteq [0, 1]$, almost all reals have *binary density* with respect to P: for almost every point x, if $x \in P$ then asymptotically, P fills up all the space around x as we "zoom in" on x; the same holds for the complement of P if $x \notin P$.

The simplest effective version of the Lebesgue density theorem is the case that P is an effectively closed set. Let us call a Martin-Löf random real z *density random* if every effectively closed set containing z has density one at z. The investigation of the Lebesgue density theorem using algorithmic randomness solved a long-standing question in algorithmic randomness known as the ML-covering problem, which was first asked by Stephan in 2004. This question was about the computational interaction of random and very-far-from-random subset of $\mathbb{N}$: is every far-from-random (i.e., K-trivial) set A computable from a Martin-Löf random set Z that does, however, not compute the halting problem? It was surprising that the affirmative answer required an analytic concept, not directly connected to the question itself: if Z is Martin-Löf random, but not density random, then Z computes the far-from-random set A.

Density randomness has been characterized in multiple ways by using effective versions of other well-known "almost everywhere" theorems due to Lebesgue; for instance, Miyabe, Nies and Zhang [19] characterized it as the points where the Lebesgue differentiation theorem holds for lower semi-computable functions. On the other hand, no characterization purely in terms of computational complexity is known; in particular, we do not know whether for a Martin-Löf random set, density randomness is equivalent to *not* computing all the far-from-random sets.

Porosity is an important geometric concept that arises frequently when one studies sets of differentiability of a function. We say that a class $P \subseteq \mathbb{R}$ is porous at a real z if arbitrary short intervals I around z have a subinterval L, of size a fixed fraction of the size of I, such that $L \cap P = \emptyset$; z is a non-porosity point if no effectively closed class P containing z is porous at z. Clearly each density random point is a non-porosity point. Bienvenu, Hölzl, Miller and Nies [8] relate porosity to complexity: a Turing incomplete random real is a non-porosity point for all effectively closed P. Many related questions, such as the converse of their result, remain open.

Combining density with the cost function method (see e.g. Nies, Computability and Randomness, [21]) has recently allowed Greenberg, Miller, and Nies to understand subclasses of the far-from random (K-trivial) sets. They proved that a c.e. set is below both halves of some Martin-Löf random oracle if and only if it obeys the cost function $c(x, s) = \sqrt{\Omega_s - \Omega_x}$, where Ω_t is the state of Chaitin's halting probability at stage t. Intuitively, obedience to a cost function c means that few numbers can be enumerated into the set, because the total cost of enumerations has to be kept finite; $c(x, s)$ defines the cost of putting x as a least element into the set at a stage $s > x$.

3 Presentation Highlights

The talks fell roughly into the areas highlighted in Section 1, though with ample crossover. We held talk review sessions at three evenings, where each speaker came to the front and was asked questions for 10-15 minutes, with the slides available. These session were popular because they helped deepen the understanding of the topics, and prompted speakers to comment on background and motivation more freely than what is standard in conference talks.

3.1 Computable analysis

From the point of view of computable analysis, Nathaniel Ackerman reported on joint work with Cameron Freer as to the extent to which Choquet theory can be understood in computational terms. Choquet's theorem states that given a closed, convex, metrizible space X and any element $x \in X$, there is a probability measure μ_x that represents x and is supported on the extreme points of X; and if X is compact, then X is a simplex if and only if for every x there is a unique such μ_x. Ackermann and Freer studied the recursion-theoretic complexity of computing the extreme points of a compact closed subset and representing measures, and so on. Similarly, in his talk, Freer discussed unique ergodicity and measures invariant under permutations of the natural numbers, from the perspective of computable analysis.

Along similar lines, Cristobal Rojas considered notions of computability in dynamical systems, specifically regarding the complexity of Mandelbrot-like fractal sets.

3.2 Algorithmic randomness

Johanna Franklin and Henry Towsner have shown the very elegant result that the class of points in an effective dynamical system at which the conclusion of the pointwise ergodic theorem always holds for a computable function is exactly the class of Martin-Löf random points. As mentioned above, characterizing these points when the function is only required to be lower-semicomputable is still an open problem, and Franklin and Towsner discussed some of the difficulties in solving this problem in their joint talk.

George Barmpalias talked about his recent result with Lewis-Pye and Teutsch on the optimal use bound in the Kucera- Gacs Theorem (which states that every set can be computed from some ML-random set). He provided excellent background, including a related question of Bennett on computational depth, and then discussed some of the technics behind their construction.

Miller (replacing a talk of a participant that cancelled) discussed co-totality of enumeration degrees. Some of the motivation stems from analysing the complexity of minimal subshifts; the connection is that the e-degree of the language of the subshift (i.e. all allowed patterns) is co-total. This work connects the research groups that have worked independently up to recently- people working in enumeration degrees and people doing subshifts, such as Jeandel and Vanier.

Monin gave an overview of recent results in higher randomness, a field where instead of algorithmic tests, one uses the stronger kinds of tests that are allowed certain "infinite computations". Monin and others have recently solved a number of open problems, such as showing that lowness for Π^1_1 randomness coincides with being hyperarithmetical.

An important component in the study of algorithmic randomness is the notion of Martin-Löf Randomness relative to a noncomputable measure. Surprisingly, the corresponding notion for Schnorr randomness had not been developed. Jason Rute filled this gap in his talk, presenting such a definition. He also posed as an open question the tasks of characterising, for any given oracles A and B, conditions under which for every measure μ, every element that is μ-Schnorr random relative to B is μ-Schnorr random relative to A. (Background: restricted to the uniform measure, with respect to Martin-Löf randomness, one obtains the so-called LR-reducibility, which is weaker than Turing reducibility $A \leq_T B$, and was introduced by Nies [20]. A characterisation in the case of Schnorr randomness and uniform measure has been provided by Miyabe [18].)

Borel's early result that almost all numbers are normal to all bases is seminal to the study of randomness, and it is natural to ask as to the randomness requirements that are sufficient to ensure that a number has this property. With a suitable formulation of the problem, one can even consider non-integer bases. Santiago Figueira reported on joint work with Javier Almarza, which shows that any polynomial-time random real is normal with respect to any *Pisot* base (and hence, in particular, with respect to any integer base greater than 1).

Similarly, Furstenberg's 1075 multiple recurrence theorem is a landmark result in ergodic theory, and, since it asserts that a property holds pointwise a.e., it is natural to ask for algorithmic randomness requirements that are sufficient to ensure that the conclusion holds, relative to the complexity of the data. Satyadev Nandakumar reported on joint work with Rod Downey and Nies (arXiv:1604.04230) that addressed questions of this sort. For example, Martin-Löf random elements are multiply recurrent with respect to non-null Π^0_1-classes.

3.3 Reverse mathematics and descriptive complexity

In his talk, Keita Yokoyama brought together the fields of computable analysis, randomness, and reverse mathematics. He discussed results by Brattka, Miller, and Nies as well as results by Greenberg, Miller, and Nies which characterize the Jordan decomposition theorem and Lebesgue's theorem on the differentiability of bounded variation functions both in terms of the recursion-theoretic complexity of the objects asserted to exist. He then presented joint work with Marcus Triplett and Nies which characterizes the strengh of these theorems in terms of reverse mathematics.

A number of talks dealt with the descriptive complexity of classes of objects that arise in ergodic theory and analysis, and reductions between them. For example, Linda Brown-Westrick showed that the class of absolutely continuous functions on the unit interval is Π^0_3 complete, and that the set of functions obtainable by Denjoy integration is a complete Π^1_1 set. Vasco Brattka considered relationships between classes that arise in algorithmic randomness in terms of Weihrauch reducibility. He also characterized two notions of random computation, Las Vegas computability and Monte Carlo computability, in terms of classes WKL and $WWKL$ that arise in reverse mathematics.

Kihara reported on joint work with Montalban on a uniform version of Martin's conjecture for m-reducibility. Consider uniformly degree invariant "Turing to many-one" operators (for instance the one induced by the usual Turing jump on sets). Under the set theoretic axiom AD^+, the ordering given by comparison on upper cones is isomorphic to the Wadge degrees. There is in fact more general version for better quasi orderings Q and Wadge degrees of Q-valued functions.

3.4 Proof mining

Avigad surveyed results on the effective content of ergodic theory, focusing on the kinds of effective information one might hope to extract from the proof of a convergence theorem in mathematics: bounds on the rate of convergence, bounds on the number of oscillations (closely related to upcrossing inequalities, in the sense of Erret Bishop), and bounds on the rate of metastability. Ulrich Kohlenbach discussed recent work in proof mining involving the $Cat(\kappa)$ spaces that played a central role in Gromov's work on geometric group theory. In particular, he showed that one can extract effective rates of asymptotic regularity and bounds on rates of metastability associated to ergodic-theoretic convergence results on $Cat(\kappa)$ spaces.

4 Scientific Progress Made

In a working session led by Brattka and Kohlenbach, participants discussed the various types of effective information that arise in computable analysis (for example, effective notions of compactness), and clarified relationships between them.

A second working group that met twice focussed on subshifts, an important notion from ergodic. An introductory short presentation by Nies exposed the ramifications of subshifts in the areas of computability, descriptive set theory, tilings, and algebra. Thereafter various participants described results they either had proved, or knew of, in these areas. E.g., Miller exposed connections to enumeration degrees, and Westrick told of her results relating to Wang tiles she obtained after recent discussions with the Marseille group around Durand. The second session focussed on open problems.

The interaction of researchers led to new collaboration and progress in various directions. We select some topics were progress was made.

Hirschfeldt and Monin discussed the problem whether there is a non-computable Δ^0_2 set such that no set of hyperimmune-free degree joins it above $\emptyset'$. Their subsequent work led to an affirmative answer.

Turetsky and Westrick made progress on the relationship of multiple recurrence and algorithmic randomness, related to the work of Downey, Nandakumar and Nies.

Franklin and Towsner advanced on the main problem left in the paper they exposed in their talk.

Rute, Brattka and Nies discussed effective topological groups. Subsequently Rute showed that for compact computable groups, *effective* compactness is equivalent to computability of the Haar measure.

Rute, in his talk at and AMS meeting and at the Capulalpan retreat, described a uniform reducibility $x \leq_r y$ in the setting of points x, Y in computable metric spaces: there is a functional Φ defined on an effectively closed set that $\Phi(y) = x$.

Kihara and Rute showed at the CMO that two spaces have the same uniform degree structure if and only if there is an effectively Δ^0_2 measurable bijection between them. Now they are researching the meaning of this concept in the framework of topological dimension theory. For instance, a point x in a computable metric space X is contained in an effectively closed set of dimension n if and only if it is uniformly equivalent to a point in M_n, the universal compact metric space of dimension n.

A 4-day retreat organised by Nies was held in Capulalpan de Mendez (two hours from Oaxaca in the Sierra) prior to the meeting. Turismo el Convento in Oaxaca (Fanny Santillan) were responsible for the logistical side. The complete funding of USD 2500 was provided in equal parts from Nies' grant and the University of Auckland. Such retreats provide and intense climate of collaboration in an interesting location, which is an excellent way for fostering collaboration and led well into the workshop. The retreat included an "ARA" meeting, in this case one day of informal talks by the 8 participants. It also resulted in interesting open questions which are posted on the 2017 Logic Blog accessible from Nies' web site.

5 Outcome of the Meeting

The retreat and meeting resulted in closer connections between researchers from different parts around the globe, and in different scientific areas. In particular, it improved the connections between researchers in proof theory/reverse mathematics and researchers in more traditional areas of computability. For instance, participants in the latter areas understand metastable convergence better now. Connections to analysis and ergodic theory pervaded the program. This had the welcome effect of making many researchers aware of interesting results and questions that stem from these interactions.

The results of the interactions at the meeting will be published in good journals and conference papers. The meeting will also impact on secondary literature. One topic of discussions between participants was the possibility to cover the results in randomness and its interactions with analysis and other fields in a new book, or significant extension of an existing book such as [21]. Barmpalias was prompted by the meeting to summarise to Nies the considerable number of open problems from [21] he has solved after its appearance; this summary will be made publicly available. In a broader context, the meeting helped pointing a way to future research into algorithmic randomness via its multiple interactions with analysis, ergodic theory, reverse mathematics and other directions. The corresponding questions were still unforeseen during the writing of the text books [21] and [13].

The meeting also helped for future meetings on related topics, both meetings in the planning stage, and ones already organised. The two meetings of the latter kind are the CCR in Bangalore, India and CCA in Daejon, Korea, both in July 2017. The presentations of these meetings at the CMO should result in higher number of submissions and higher attendance.

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