Probabilistic Computability and Randomness in the Weihrauch Lattice

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Outline

1. The Weihrauch Lattice
2. Vitali Covering Theorem
3. Las Vegas and Monte Carlo Computability
The Weihrauch Lattice
Weihrauch Reducibility

Consider $f : \subseteq X \Rightarrow Y$ and $g : \subseteq Z \Rightarrow W$.

- $f$ is Weihrauch reducible to $g$, $f \leq_W g$, if there are computable $H : \subseteq X \times W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $H(\text{id}_X, gK) \sqsubseteq f$.
- $f$ is strongly Weihrauch reducible to $g$, $f \leq_{sW} g$, if there are computable $H : \subseteq W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $HgK \sqsubseteq f$.
- Equivalences $f \equiv_W g$ and $f \equiv_{sW} g$ are defined as usual.
Consider $f : \subseteq X \Rightarrow Y$ and $g : \subseteq Z \Rightarrow W$.

$f$ is **Weihrauch reducible** to $g$, $f \leq_W g$, if there are computable $H : \subseteq X \times W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $H(id_X, gK) \sqsubseteq f$.

$f$ is **strongly Weihrauch reducible** to $g$, $f \leq_{sW} g$, if there are computable $H : \subseteq W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $HgK \sqsubseteq f$.

**Equivalences** $f \equiv_W g$ and $f \equiv_{sW} g$ are defined as usual.
Weihrauch Reducibility

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- $f$ is strongly Weihrauch reducible to $g$, $f \leq_{sW} g$, if there are computable $H : \subseteq W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $HgK \sqsubseteq f$.
- Equivalences $f \equiv_W g$ and $f \equiv_{sW} g$ are defined as usual.
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$\exists f$ is Weihrauch reducible to $g$, $f \leq_{W} g$, if there are computable $H : \subseteq X \times W \Rightarrow Y$, $K : \subseteq X \Rightarrow Z$ such that $H(id_X, gK) \sqsubseteq f$.

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Equivalences $f \equiv_{W} g$ and $f \equiv_{sW} g$ are defined as usual.
Examples of Mathematical Problems

▶ The **Limit Problem** is the mathematical problem

\[ \lim : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, \ldots \rangle \mapsto \lim_{i \rightarrow \infty} p_i \]

with \( \text{dom}(\lim) := \{ \langle p_0, p_1, \ldots \rangle : (p_i)_i \text{ is convergent} \} \).

▶ **Martin-Löf Randomness** is the mathematical problem

\[ \text{MLR} : 2^\mathbb{N} \Rightarrow 2^\mathbb{N} \text{ with} \]

\[ \text{MLR}(x) := \{ y \in 2^\mathbb{N} : y \text{ is Martin-Löf random relative to } x \} \].

▶ **Weak Weak König's Lemma** is the mathematical problem

\[ \text{WWKL} : \subseteq \text{Tr} \Rightarrow 2^\mathbb{N}, T \mapsto [T] \]

with \( \text{dom}(\text{WWKL}) := \{ T \in \text{Tr} : \mu([T]) > 0 \} \).

▶ The **Intermediate Value Theorem** is the problem

\[ \text{IVT} : \subseteq \text{Con}[0, 1] \Rightarrow [0, 1], f \mapsto f^{-1}\{0\} \]

with \( \text{dom}(\text{IVT}) := \{ f : f(0) \cdot f(1) < 0 \} \).

▶ The **Choice Problem** \( C_X : \subseteq A(X) \Rightarrow X, A \mapsto A \).

\( \text{PC}_X \) is \( C_X \) restricted to sets \( A \) with \( \mu(A) > 0 \).
Examples of Mathematical Problems

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- **The Choice Problem** \( \text{C}_X : \subseteq \mathcal{A}X \Rightarrow X, A \mapsto A \).
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- **The Choice Problem** \( C_X : \mathbb{N}^\mathbb{N} \Rightarrow [X, A \mapsto A] \)
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Algebraic Operations

Definition

For \( f : \subseteq X \Rightarrow Y \) and \( g : \subseteq W \Rightarrow Z \) we define:

- \( f \times g : \subseteq X \times W \Rightarrow Y \times Z, \ (x, w) \mapsto f(x) \times g(w) \) (Product)
- \( f \sqcup g : \subseteq X \sqcup W \Rightarrow Y \sqcup Z, \ z \mapsto \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in W \end{cases} \) (Coproduct)
- \( f \sqcap g : \subseteq X \times W \Rightarrow Y \sqcup Z, \ (x, w) \mapsto f(x) \sqcap g(w) \) (Sum)
- \( f^* : \subseteq X^* \Rightarrow Y^*, f^* = \bigsqcup_{i=0}^{\infty} f^i \) (Star)
- \( \hat{f} : \subseteq X^\mathbb{N} \Rightarrow Y^\mathbb{N}, \ \hat{f} = \bigcup_{i=0}^{\infty} f^i \) (Parallelization)

- Weihrauch reducibility induces a lattice with the coproduct \( \sqcup \) as supremum and the sum \( \sqcap \) as infimum.
- Parallelization and star operation are closure operators in the Weihrauch lattice.
Algebraic Operations

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For \( f : \subseteq X \rightrightarrows Y \) and \( g : \subseteq W \rightrightarrows Z \) we define:

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- \( f \sqcap g : \subseteq X \times W \rightrightarrows Y \sqcup Z \), \( (x, w) \mapsto f(x) \sqcap g(w) \) (Sum)
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Basic Complexity Classes and Reverse Mathematics

\[ \begin{align*}
C_{NN} & \equiv_{sW} \hat{C}_N \\
\lim & \equiv_{sW} \hat{C}_N \\
C_R & \equiv_{sW} C_N \times C_{2N} \\
WKL & \equiv_{sW} C_{2N} \equiv_{sW} \hat{C}_2 \\
WWKL & \equiv_{sW} PC_{2N} \\
K_N & \equiv_{sW} C_2^* \\
C_1 & \\
\text{ATR}_0 & \\
\text{ACA}_0 & \\
\text{WWKL}_0 + I\Sigma^0_1 & \\
\text{WKL}_0 & \\
\text{WWKL}_0 & \\
I\Sigma^0_1 & \\
\text{BS}_1^0 & \\
\text{RCA}_0 &
\end{align*} \]
Quantitative Versions of WWKL

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

By $\varepsilon$-WWKL $\subseteq \text{Tr} \Rightarrow 2^{\mathbb{N}}$ we denote the restriction of WKL to $\text{dom}(\varepsilon$-WWKL) := $\{T : \mu([T]) > \varepsilon\}$ for $\varepsilon \in \mathbb{R}$.

Theorem (DDHMS 2016 and B., Gherardi and Hölzl 2015)

$\varepsilon$-WWKL $\leq_{W} \delta$-WWKL $\iff \varepsilon \geq \delta$ for all $\varepsilon, \delta \in [0, 1]$.

Proof. (Idea) $\Rightarrow$ Assume $\varepsilon < \delta$. Then there are positive integers $a, b$ with $\varepsilon < \frac{a}{b} \leq \delta$. We consider

- $C_{a,b}$ which is $C_{b}$ restricted to sets $A \subseteq \{0, ..., b - 1\}$ with $|A| \geq a$.

Then $C_{a,b} \leq_{W} \varepsilon$-WWKL and $C_{a,b} \not\leq_{W} \delta$-WWKL. Hence $\varepsilon$-WWKL $\not\leq_{W} \delta$-WWKL.

The separation is purely topological, i.e., Weihrauch reducibility can be replaced by its continuous counterpart.
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Weak Weak Kőnig’s Lemma - The Uniform Scenario

\[ \lim \]
\[ C_R \equiv_{sW} WKL \times C_N \]
\[ C_{2N} \equiv_{sW} WKL \]
\[ PC_R \equiv_{sW} WWKL \times C_N \rightarrow C_N \ast MLR \]
\[ PC_{2N} \equiv_{sW} WWKL \]
\[ \ast- WWKL \]
\[ \vdots \]
\[ \frac{1}{k+1}- WWKL \]
\[ \frac{1}{k}- WWKL \]
\[ \vdots \]
\[ (1 - \ast)- WWKL \]
\[ MLR \equiv_W (C_N \rightarrow WWKL) \]
The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

**Definition**

For two mathematical problem \( f, g \) we define

\[
\begin{align*}
\triangleright f \ast g & := \max \{ f_0 \circ g_0 : f_0 \leq_W f, \ g_0 \leq_W g \} \quad \text{compos. product} \\
\triangleright g \rightarrow f & := \min \{ h : f \leq_W g \ast h \} \quad \text{implication}
\end{align*}
\]

**Theorem (B. and Pauly 2016)**

The compositional product \( f \ast g \) and the implication \( g \rightarrow f \) exist for all problems \( f, g \).
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### Definition

For two mathematical problem $f, g$ we define

- $f \ast g := \max\{f_0 \circ g_0 : f_0 \leq_W f, g_0 \leq_W g\}$ \textit{compos. product}
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### Theorem (B. and Pauly 2016)

*The compositional product $f \ast g$ and the implication $g \rightarrow f$ exist for all problems $f, g$.***
Proposition (B. and Pauly 2016)

\[ \text{MLR} \equiv_W (C_N \rightarrow \text{WWKL}). \]

**Proof.** \((C_N \rightarrow \text{WWKL}) \leq_W \text{MLR}: \) It suffices to prove \(\text{WWKL} \leq_W C_N \ast \text{MLR},\) which follows from Kučera’s Lemma.

\(\text{MLR} \leq_W (C_N \rightarrow \text{WWKL}): \) Given some \(h\) with \(\text{WWKL} \leq_W C_N \ast h\) we need to prove that \(\text{MLR} \leq_W h.\) Given some universal Martin-Löf test \((U_i);\), we use \(A_0 := 2^\mathbb{N} \setminus U_0\) and the fact that Martin-Löf randoms are stable under finite changes. □

Proposition (B., Gherardi and Hölzl 2015)

\[ \text{MLR} \ast \text{MLR} \leq_W \text{MLR} \]

**Proof.** This is a consequence of van Lambalgen’s Theorem. □

Corollary

The class of functions \(f \leq_W \text{MLR}\) is closed under composition.
Martin-Löf Randomness

Proposition (B. and Pauly 2016)

\[ \text{MLR} \equiv W(C_N \rightarrow \text{WWKL}). \]

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\(\text{MLR} \leq_W (C_N \rightarrow \text{WWKL}): \) Given some \(h\) with \(\text{WWKL} \leq_W C_N \ast h\) we need to prove that \(\text{MLR} \leq_W h. \) Given some universal Martin-Löf test \((U_i)_i, \) we use \(A_0 := 2^\mathbb{N} \setminus U_0\) and the fact that Martin-Löf randoms are stable under finite changes. □

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**Corollary**

The class of functions \(f \leq_W \text{MLR} \) is closed under composition.
### Proposition (B. and Pauly 2016)

\[ \text{MLR} \equiv \text{W}(C_N \rightarrow \text{WWKL}). \]

**Proof.**  
\((C_N \rightarrow \text{WWKL}) \leq_{W} \text{MLR}:\) It suffices to prove \(\text{WWKL} \leq_{W} C_N \ast \text{MLR},\) which follows from Kučera’s Lemma.  

\(\text{MLR} \leq_{W} (C_N \rightarrow \text{WWKL}):\) Given some \(h\) with \(\text{WWKL} \leq_{W} C_N \ast h\) we need to prove that \(\text{MLR} \leq_{W} h.\) Given some universal Martin-Löf test \((U_i)_i,\) we use \(A_0 := 2^\mathbb{N} \setminus U_0\) and the fact that Martin-Löf randoms are stable under finite changes.  

### Proposition (B., Gherardi and Hölzl 2015)

\[ \text{MLR} \ast \text{MLR} \leq_{W} \text{MLR} \]

**Proof.** This is a consequence of van Lambalgen’s Theorem.  

### Corollary

The class of functions \(f \leq_{W} \text{MLR}\) is closed under composition.
**Proposition (B. and Pauly 2016)**

\[ \text{MLR} \equiv_W (C_N \to \text{WWKL}). \]

**Proof.** \((C_N \to \text{WWKL}) \leq_W \text{MLR}: \) It suffices to prove \(\text{WWKL} \leq_W C_N \ast \text{MLR},\) which follows from Kučera’s Lemma.

\(\text{MLR} \leq_W (C_N \to \text{WWKL}): \) Given some \(h\) with \(\text{WWKL} \leq_W C_N \ast h\) we need to prove that \(\text{MLR} \leq_W h.\) Given some universal Martin-Löf test \((U_i)_i,\) we use \(A_0 := 2^\mathbb{N} \setminus U_0\) and the fact that Martin-Löf randoms are stable under finite changes. \(\square\)

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The class of functions \(f \leq_W \text{MLR}\) is closed under composition.
Proposition (B. and Pauly 2016)

\[ \text{MLR} \equiv W(C_N \to WWKL). \]

Proof. \((C_N \to WWKL) \leq W \text{ MLR}: \) It suffices to prove \(WWKL \leq W C_N \ast \text{MLR},\) which follows from Kučera’s Lemma.

\(\text{MLR} \leq W(C_N \to WWKL): \) Given some \(h\) with \(WWKL \leq W C_N \ast h\) we need to prove that \(\text{MLR} \leq W h.\) Given some universal Martin-Löf test \((U_i)_i,\) we use \(A_0 := 2^\mathbb{N} \setminus U_0\) and the fact that Martin-Löf randoms are stable under finite changes. □

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Proof. This is a consequence of van Lambalgen’s Theorem. □

Corollary

The class of functions \(f \leq W \text{ MLR}\) is closed under composition.
**Definition**

The jump $f' : \subseteq X \rightarrow Y$ of $f : \subseteq X \rightarrow Y$ is the same problem, but with the input representation $\delta$ of $X$ replaced by $\delta' := \delta \circ \text{lim}$. 

A name of an object $x \in X$ with respect to $\delta'$ is a sequence that converges to a name with respect to $\delta$. Examples:

- $\text{id}' =_{sW} \text{lim}, \ WKL' =_{sW} \text{KL} =_{sW} \text{BWT}_R, \ n\text{-RAN} =_{sW} \text{MLR}^{(n-1)}$.

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Jump Inversion

Theorem (B., Hölzil and Kuyper 2016)

1. \( f' \leq_W g' \) relative to \( p \) \( \implies \) \( f \leq_W g \) relative to \( p' \).
2. \( f' \leq_{sW} g' \) relative to \( p \) \( \implies \) \( f \leq_{sW} g \) relative to \( p' \).


If there exist a continuous \( F \) such that the diagram commutes, then \( G \) is continuous. \( \square \)
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**Proof.** Jump Control Theorem (B., Hendtlass and Kreuzer 2015):

If there exist a continuous \( F \) such that the diagram commutes, then \( G \) is continuous.
Theorem (B., Hölzl and Kuyper 2016)

1. $f' \leq_W g'$ relative to $p$ $\implies$ $f \leq_W g$ relative to $p'$.
2. $f' \leq_{sW} g'$ relative to $p$ $\implies$ $f \leq_{sW} g$ relative to $p'$.


If there exist $F$ computable relative to $p$ such that the diagram commutes, then $G$ is computable relative to $p'$.

\qed
Weak Weak König’s Lemma - Jumps (work in progress)

\[ \lim_{n \to \infty} (C_{2^n}^{(n)}) \equiv_{sW} WKL^{(n)} \times C^{(n)}_N \]

\[ C^{(n)}_2 \equiv_{sW} WKL^{(n)} \]

\[ PC^{(n)}_R \equiv_{sW} WWKL^{(n)} \times C^{(n)}_N \]

\[ PC^{(n)}_{2^n} \equiv_{sW} WWKL^{(n)} \]

\[ (*)-WWKL^{(n)} \]

\[ \frac{1}{k+1}-WWKL^{(n)} \]

\[ \frac{1}{k}-WWKL^{(n)} \]

\[ (1-\ast)-WWKL^{(n)} \]

\[ \Sigma^0_{n+2} \text{-measurable} \]
Further Notions of Randomness

Theorem (Hölzl and Miyabe 2015)

\[ \text{WR} <_W \text{SR} <_W \text{CR} <_W \text{MLR} <_W \text{W2R} <_W 2\text{-RAN}. \]

**Proof.** The strictness has been proved using hyperimmune degrees, high degrees and minimal degrees.

- **WR**: Kurtz random
- **SR**: Schnorr random
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- **W2R**: weakly 2-random
- **n-RAN**: \( n \)-random

**Question**

*Find other characterizations of randomness notions \( R \) of the form \( R \equiv_W (A \to B) \), e.g., 1-GEN \( \equiv_W (? \to \text{BCT}_0') \).*
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Theorem of Kurtz. Every 2–random computes a 1–generic.

Theorem (B., Hendtlass and Kreuzer 2015)

\[1\text{-GEN} \lesssim_w 2\text{-RAN}.\]

Proof. (Idea) We apply the “fireworks technique” of Rumyantsev and Shen to get a uniform reduction. \(\square\)

Theorem (B., Hendtlass and Kreuzer 2015)

\[\text{BCT}_0' \not\lesssim_w \text{WWKL}^{(n)} \text{ for all } n \in \mathbb{N}.\]

Proof. (Idea) There exists a co-c.e. comeager set \(A \subseteq 2^\mathbb{N}\) such that no point of \(A\) is low for \(\Omega\). \(\text{WWKL}^{(n)}\) has a realizer that maps computable inputs to outputs that are low for \(\Omega\) for \(n \geq 1\). \(\square\)

Corollary

\[\text{BCT}_0' \not\lesssim_w 1\text{-GEN}.\]
Uniform Theorem of Kurtz

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Theorem (B., Hendtlass and Kreuzer 2015)

\[ \text{1-GEN} \preceq_{\text{W}} \text{2-RAN}. \]

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Theorem (B., Hendtlass and Kreuzer 2015)

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**Corollary**

\[ B\text{CT}^\prime_0 \nleq_W 1\text{-GEN}. \]
Vitali Covering Theorem
A point $x \in \mathbb{R}$ is captured by a sequence $\mathcal{I} = (I_n)_n$ of open intervals, if for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ with $\text{diam}(I_n) < \varepsilon$ and $x \in I_n$.

$\mathcal{I}$ is a Vitali cover of $A \subseteq \mathbb{R}$, if every $x \in A$ is captured by $\mathcal{I}$.

$\mathcal{I}$ eliminates $A$, if the $I_n$ are pairwise disjoint and $\lambda(A \setminus \bigcup \mathcal{I}) = 0$ (where $\lambda$ denotes the Lebesgue measure).

**Theorem (Vitali Covering Theorem)**

*If $\mathcal{I}$ is a Vitali cover of $[0, 1]$, then there exists a subsequence $\mathcal{J}$ of $\mathcal{I}$ that eliminates $[0, 1]$.***
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Theorem (Brown, Giusto and Simpson 2002)

Over $\textbf{RCA}_0$ the Vitali Covering Theorem is equivalent to Weak Weak Kőnig’s Lemma $\textbf{WWKL}_0$.

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Theorem (Diener and Hedin 2012)

Using intuitionistic logic (and countable and dependent choice) the Vitali Covering Theorem is equivalent to Weak Weak Kőnig’s Lemma $\textbf{WWKL}$. 
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Vitali Covering Theorem

- $\mathcal{I}$ is called saturated, if $\mathcal{I}$ is a Vitali cover of $\bigcup \mathcal{I} = \bigcup_{n=0}^{\infty} I_n$.

Definition (Contrapositive versions of the Vitali Covering Theorem)

- **VCT$_0$**: Given a Vitali cover $\mathcal{I}$ of $[0, 1]$, find a subsequence $\mathcal{J}$ of $\mathcal{I}$ that eliminates $[0, 1]$.
- **VCT$_1$**: Given a saturated $\mathcal{I}$ that does not admit a subsequence that eliminates $[0, 1]$, find a point that is not covered by $\mathcal{I}$.
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- **VCT$_0$** : $(S \land C) \rightarrow E$,
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**Theorem (B., Gherardi, Hölzl and Pauly 2016)**

- **$\text{VCT}_0$** is computable,
- **$\text{VCT}_1 \equiv_{sW} \text{WWKL}$**, 
- **$\text{VCT}_2 \equiv_{sW} \text{WWKL} \times \mathbb{C}_\mathbb{N}$**.
**Vitali Covering Theorem**

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**Theorem (B., Gherardi, Hölzl and Pauly 2016)**

- \( \mathbf{VCT}_0 \) **is computable**,  
- \( \mathbf{VCT}_1 \equiv_{SW} \text{WWKL} \),  
- \( \mathbf{VCT}_2 \equiv_{SW} \text{WWKL} \times \mathbb{C}_\mathbb{N} \).
Proof.

- The proof of computability of $\text{VCT}_0$ is based on a construction that repeats steps of the classical proof of the Vitali Covering Theorem (and is not just based on a waiting strategy).
- The proof of $\text{VCT}_1 \equiv_{\text{sW}} \text{WWKL}$ is based on the equivalence chain $\text{VCT}_1 \equiv_{\text{sW}} \text{PC}_{[0,1]} \equiv_{\text{sW}} \text{WWKL}$.
- We use a Lemma by Brown, Giusto and Simpson on “almost Vitali covers” in order to prove $\text{VCT}_2 \leq_{\text{sW}} \text{WWKL} \times C_N$. The harder direction is the opposite one for which it suffices to show $C_N \times \text{VCT}_2 \leq_{\text{sW}} \text{VCT}_2$ by an explicit construction:
Vitali Covering Theorem in the Weihrauch Lattice

\[ \mathcal{C}_R \equiv_{SW} \text{WKL} \times \mathcal{C}_N \]

\[ \text{HBT}_1 \equiv_{SW} \mathcal{C}_{[0,1]} \equiv_{SW} \text{WKL} \]

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\[ \text{VCT}_1 \equiv_{SW} \text{PC}_{[0,1]} \equiv_{SW} \text{WWKL} \]

\[ \text{ACT} \equiv_{SW} \ast\text{-WWKL} \]

\[ \text{VCT}_0 \]

\[ \text{ACT} : \text{Int} \Rightarrow [0,1], \mathcal{I} \mapsto [0,1] \setminus \bigcup \mathcal{I}, \text{ where } \text{dom(\text{ACT})} \text{ is the set of all non-disjoint } \mathcal{I} = (I_n)_n \text{ with } \sum_{n=0}^{\infty} \lambda(I_n) < 1. \]
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Vitali Covering Theorem in the Weihrauch Lattice

$C_\mathbb{R} \equiv_{SW} \text{WKL} \times C_\mathbb{N}$

$HBT_1 \equiv_{SW} C_{[0,1]} \equiv_{SW} \text{WKL}$

$VCT_2 \equiv_{SW} PC_\mathbb{R} \equiv_{SW} \text{WWKL} \times C_\mathbb{N}$

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$VCT_0 \equiv_{SW} *\text{-WWKL}$

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$\bigcup \mathcal{I}$,
Las Vegas and Monte Carlo
Computability
Non-Deterministic Turing Machines

$$\setminus$$

Condition: $$(\forall x \in \text{dom}(f)) \{r \in R : r \text{ does not fail with } x\} \neq \emptyset$$
Las Vegas Turing Machines

Condition: \((\forall x \in \text{dom}(f)) \ \mu \{ r \in R : r \text{ does not fail with } x \} > 0\)
Monte Carlo Turing Machines

**Condition:** \((\forall x \in \text{dom}(f)) \mu\{r \in R : r \text{ does not fail with } x\} > 0\)
Non-Deterministic Computability

Proposition (B., de Brecht and Pauly 2012)

\[ f \leq_w WKL \iff f \text{ is non-deterministically computable.} \]

Non-deterministically computable functions (in this model) were first introduced and studied by Martin Ziegler.

Theorem (Gherardi and Marcone 2009)

The class of \( f \leq_w WKL \) is closed under composition.

There are at least three independent proofs:
- The original proof in terms of the separation problem.
- A proof by B. and Gherardi in terms of Kleene’s ternary logic.
- A very simple proof in terms of non-deterministically computable functions by B., de Brecht and Pauly.

Corollary

\[ WKL \equiv_w WKL \ast WKL. \]
Proposition (B., de Brecht and Pauly 2012)

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- The original proof in terms of the separation problem.
- A proof by B. and Gherardi in terms of Kleene’s ternary logic.
- A very simple proof in terms of non-deterministically computable functions by B., de Brecht and Pauly.

Corollary

\[ \text{WKL} \equiv_{\text{W}} \text{WKL} \ast \text{WKL}. \]
Proposition (B., de Brecht and Pauly 2012)

\( f \leq_W \text{WKL} \iff f \text{ is non-deterministically computable.} \)

Non-deterministically computable functions (in this model) were first introduced and studied by Martin Ziegler.

Theorem (Gherardi and Marcone 2009)

The class of \( f \leq_W \text{WKL} \) is closed under composition.

There are at least three independent proofs:

- The original proof in terms of the separation problem.
- A proof by B. and Gherardi in terms of Kleene’s ternary logic.
- A very simple proof in terms of non-deterministically computable functions by B., de Brecht and Pauly.

Corollary

\( \text{WKL} \equiv_W \text{WKL} \ast \text{WKL} \).
Las Vegas Computability

Proposition (B., Gherardi and Hölzl 2015)

\[ f \leq_W \text{WWKL} \iff f \text{ is Las Vegas computable.} \]

Proposition

\[ \text{WWKL} \equiv_W \text{WWKL} \ast \text{WWKL}. \]

Can be proved as for \text{WKL} in terms of Las Vegas computable functions with an additional application of Fubini’s Theorem.

Corollary

Las Vegas computable functions are closed under composition.
**Proposition** (B., Gherardi and Hölzl 2015)

\[ f \leq_{W} \text{WWKL} \iff f \text{ is Las Vegas computable.} \]

**Proposition**

\[ \text{WWKL} \equiv_{W} \text{WWKL} \ast \text{WWKL}. \]

Can be proved as for \text{WKL} in terms of Las Vegas computable functions with an additional application of Fubini’s Theorem.

**Corollary**

*Las Vegas computable functions are closed under composition.*
Proposition (B., Gherardi and Hölzl 2015)

\[ f \leq_{W} \text{WWKL} \iff f \text{ is Las Vegas computable.} \]

Proposition

\[ \text{WWKL} \equiv_{W} \text{WWKL} \ast \text{WWKL}. \]

Can be proved as for \text{WKL} in terms of Las Vegas computable functions with an additional application of Fubini’s Theorem.

Corollary

\[ \text{Las Vegas computable functions are closed under composition.} \]
Proposition (B., Hölzl and Kuyper 2016)

\[ f \leq_{\text{W}} \text{PC}_R' \equiv_{\text{W}} \text{WWKL}' \times C_N' \iff f \text{ is Monte Carlo computable.} \]

This result is based on a classification of positive \( G_\delta \)--choice by B., Hölzl, Nobrega and Pauly.

Theorem (Bienvenu and Kuyper 2016)

\[ \text{WWKL}' \ast \text{WWKL}' \equiv_{\text{W}} \text{PC}_{2N}' \ast \text{PC}_{2N}' \equiv_{\text{W}} \text{PC}_R' \ast \text{PC}_R' \equiv_{\text{W}} \text{WKL}'. \]

This contrasts \( \text{WKL}' \ast \text{WKL}' \equiv_{\text{W}} \text{WKL}'' \).

Corollary

Monte Carlo computable functions are closed under composition.
### Proposition (B., Hölzl and Kuyper 2016)

\[ f \leq^W \text{PC}_R' \equiv^W \text{WWKL}' \times C'_N \iff f \text{ is Monte Carlo computable.} \]

This result is based on a classification of positive \( G_\delta \)-choice by B., Hölzl, Nobrega and Pauly.

### Theorem (Bienvenu and Kuyper 2016)

\[ \text{WWKL}' \ast \text{WWKL}' \equiv^W \text{PC}_{2N}' \ast \text{PC}_{2N}' \equiv^W \text{PC}_R' \ast \text{PC}_R' \equiv^W \text{PC}_R'. \]

This contrasts \( \text{WKL}' \ast \text{WKL}' \equiv^W \text{WKL}'' \).

### Corollary

Monte Carlo computable functions are closed under composition.
Proposition (B., Hölzl and Kuyper 2016)

\[ f \leq_W PC'_R \equiv_W WWKL' \times C'_N \iff f \text{ is Monte Carlo computable.} \]

This result is based on a classification of positive $G_\delta$–choice by B., Hölzl, Nobrega and Pauly.

Theorem (Bienvenu and Kuyper 2016)

\[ WWKL' \ast WWKL' \equiv_W PC'_{2N} \ast PC'_{2N} \equiv_W PC'_R \ast PC'_R \equiv_W PC'_R. \]

This contrasts $WKL' \ast WKL' \equiv_W WKL''$. 

Corollary

Monte Carlo computable functions are closed under composition.
Classes of Computability

Non-deterministic
- WKL
- fixed points

Monte Carlo
- PC\(_R\) \(\equiv_{SW} \) WWKL\(^\prime\) \(\times C'_N\)
- sorting

Las Vegas
- WWKL
- Nash equilibria

IVT
- zeros

differentiation

\(\lim\)
Definition

\( \text{SORT}_n : \{0, 1, ..., n - 1\}^\mathbb{N} \rightarrow \{0, 1, ..., n - 1\}^\mathbb{N} \) is defined by

\[
\text{SORT}_n(p) := 0^{k_0}1^{k_1}...(m - 1)^{k_{m-1}} \hat{m}
\]

if \( m < n \) is the smallest digit that appears infinitely often in \( p \) and each digit \( i < m \) appears exactly \( k_i \) times in \( p \).

Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)

- \( C_N \leq_{sW} \text{SORT}_2 \leq_{sW} C'_N \)
- \( \text{IVT} \leq_{sW} \text{SORT}_2 \leq_{sW} \text{WWKL}' \)
**Sorting**

**Definition**

\( \text{SORT}_n : \{0, 1, \ldots, n - 1\}^\mathbb{N} \to \{0, 1, \ldots, n - 1\}^\mathbb{N} \) is defined by

\[
\text{SORT}_n(p) := 0^{k_0}1^{k_1}(m - 1)^{k_{m-1}}\hat{m}
\]

if \( m < n \) is the smallest digit that appears infinitely often in \( p \) and each digit \( i < m \) appears exactly \( k_i \) times in \( p \).

\[
\begin{array}{cccccccccccccccc}
0 & 3 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 3 & \\
\end{array}
\]

\[
\downarrow \text{SORT}_5
\]

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots
\end{array}
\]

**Proposition** (Neumann and Pauly, B., Höltzl and Kuyper 2016)

- \( C_N \leq_{sW} \text{SORT}_2 \leq_{sW} C'_N \)
- \( \text{IVT} \leq_{W} \text{SORT}_2 \leq_{W} \text{WWKL}' \)
Definition

\(\text{SORT}_n : \{0, 1, \ldots, n-1\}^\mathbb{N} \to \{0, 1, \ldots, n-1\}^\mathbb{N}\) is defined by

\[
\text{SORT}_n(p) := 0^{k_0}1^{k_1}(m-1)^{k_{m-1}}\hat{m}
\]

if \(m < n\) is the smallest digit that appears infinitely often in \(p\) and each digit \(i < m\) appears exactly \(k_i\) times in \(p\).

Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)

- \(C_{\mathbb{N}} \leq_{sW} \text{SORT}_2 \leq_{sW} C'_{\mathbb{N}}\)
- \(\text{IVT} \leq_{W} \text{SORT}_2 \leq_{W} \text{WWKL}'\)
Sorting in the Weihrauch Lattice

\[
\begin{align*}
\text{lim}' & \quad \text{PC}'_R \equiv_{\text{sw}} \text{WWKL}' \times C'_N \\
\text{PC}'_2^N & \equiv_{\text{sw}} \text{WWKL}' \\
C'_N \quad & \\
\text{C}'_N & \quad \text{K}'_N \\
\text{C}'_N & \quad \text{ SORT}_{n+2} \\
\text{PC}'_2^N & \equiv_{\text{sw}} \text{WWKL}' \\
\text{CC}_{[0,1]} & \equiv_{\text{sw}} \text{ IVT} \\
\Sigma^0_2 \text{– measurable} \\
\Sigma^0_3 \text{– measurable}
\end{align*}
\]
Besides COH sorting is the only problem that we know that is low\textsubscript{2} but not low in the following sense.

**Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)**

\[
\lim \ast \lim \ast \text{SORT}_2 \leq \text{W lim} \ast \lim \text{ and } \lim \ast \text{SORT}_2 \not\leq \text{W lim}.
\]

Neumann and Pauly proved that \text{SORT}_2^\ast characterizes the class of functions computable by certain algebraic machine models.

**Corollary**

\textit{BSS computable functions }\textit{f : }\mathbb{R}^\ast \rightarrow \mathbb{R}^\ast \textit{ are computable on Monte Carlo machines.}
Besides **COH** sorting is the only problem that we know that is low\textsuperscript{2} but not low in the following sense.

**Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)**

\[
\lim \ast \lim \ast \text{SORT}_2 \leq_W \lim \ast \lim \text{ and } \lim \ast \text{SORT}_2 \nleq_W \lim.
\]

Neumann and Pauly proved that \text{SORT}_2^* characterizes the class of functions computable by certain algebraic machine models.

**Corollary**

*BSS computable functions* \( f : \mathbb{R}^* \rightarrow \mathbb{R}^* \) *are computable on Monte Carlo machines.*
Besides **COH** sorting is the only problem that we know that is low\(_2\) but not low in the following sense.

**Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)**

\[
\lim * \lim * \text{SORT}_2 \leq_W \lim * \lim \text{ and } \lim * \text{SORT}_2 \not\leq_W \lim.
\]

Neumann and Pauly proved that **SORT\(_2^*\)** characterizes the class of functions computable by certain algebraic machine models.

**Corollary**

*BSS computable functions* \( f : \mathbb{R}^* \rightarrow \mathbb{R}^* \) are computable on Monte Carlo machines.
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