Probabilistic Computability and Randomness in the Weihrauch Lattice

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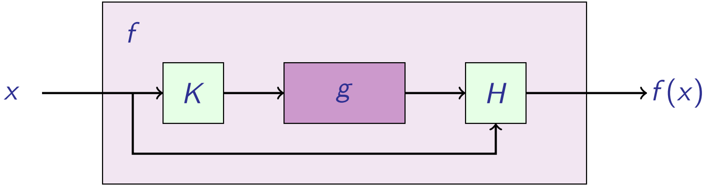
Algorithmic Randomness Interacts with Analysis and Ergodic Theory
Oaxaca, Mexico, 4–9 December 2016

Outline

The Weihrauch Lattice

Weihrauch Reducibility

Consider $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$.



- f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $H(\mathrm{id}_X, gK) \sqsubseteq f$.
- f is strongly Weihrauch reducible to g, $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $HgK \sqsubseteq f$.
- Equivalences $f \equiv_{\mathrm{W}} g$ and $f \equiv_{\mathrm{sW}} g$ are defined as usual.

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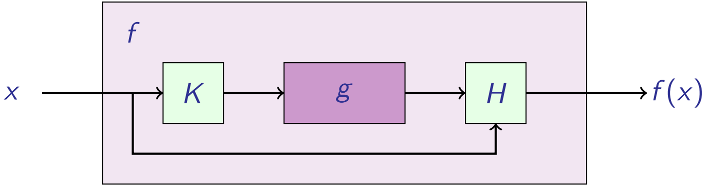

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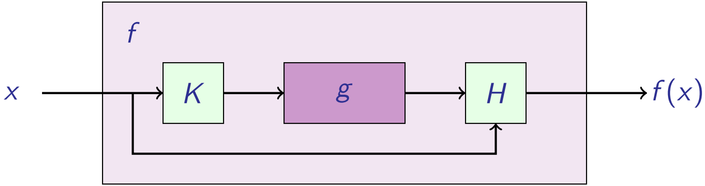

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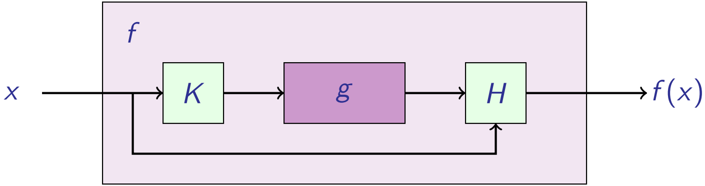

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Examples of Mathematical Problems

- The Limit Problem is the mathematical problem

$$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, \ldots \rangle \mapsto \lim_{i\to\infty} p_i$$

  with $\mathrm{dom}(\lim) := \{\langle p_0, p_1, \ldots\rangle : (p_i)_i \text{ is convergent}\}$.

- Martin-Löf Randomness is the mathematical problem $\mathsf{MLR} : 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ with

$$\mathsf{MLR}(x) := \{y \in 2^{\mathbb{N}} : y \text{ is Martin-Löf random relative to } x\}.$$

- Weak Weak Kőnig's Lemma is the mathematical problem

$$\mathsf{WWKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

  with $\mathrm{dom}(\mathsf{WWKL}) := \{T \in \mathrm{Tr} : \mu([T]) > 0\}$.

- The Intermediate Value Theorem is the problem

$$\mathsf{IVT} :\subseteq \mathrm{Con}[0,1] \rightrightarrows [0,1], f \mapsto f^{-1}\{0\}$$

  with $\mathrm{dom}(\mathsf{IVT}) := \{f : f(0) \cdot f(1) < 0\}$.

- The Choice Problem $\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$. PC_X is C_X restricted to sets A with $\mu(A) > 0$.

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Algebraic Operations

Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z,\ (x, w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z,\ z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z,\ (x, w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*,\ f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}},\ \widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (Parallelization)

- Weihrauch reducibility induces a lattice with the coproduct $\sqcup$ as supremum and the sum $\sqcap$ as infimum.
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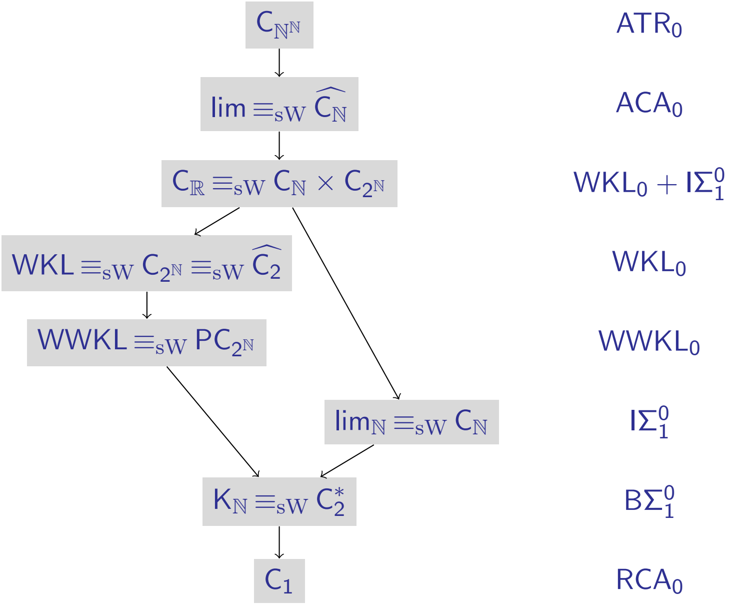
Basic Complexity Classes and Reverse Mathematics



The Probabilistic Landscape

Quantitative Versions of WWKL

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

By ε-WWKL $:\subseteq \mathsf{Tr} \rightrightarrows 2^{\mathbb{N}}$ we denote the restriction of WKL to $\mathrm{dom}(\varepsilon\text{-WWKL}) := \{T : \mu([T]) > \varepsilon\}$ for $\varepsilon \in \mathbb{R}$.

Theorem (DDHMS 2016 and B., Gherardi and Hölzl 2015)

ε-WWKL $\leq_{\mathrm{W}}$ δ-WWKL $\iff$ $\varepsilon \geq \delta$ for all $\varepsilon, \delta \in [0,1]$.

Proof. (Idea) "$\Longrightarrow$" Assume $\varepsilon < \delta$. Then there are positive integers a, b with $\varepsilon < \frac{a}{b} \leq \delta$. We consider

- $\mathsf{C}_{a,b}$ which is C_b restricted to sets $A \subseteq \{0, ..., b-1\}$ with $|A| \geq a$.

Then $\mathsf{C}_{a,b} \leq_{\mathrm{W}}$ ε-WWKL and $\mathsf{C}_{a,b} \not\leq_{\mathrm{W}}$ δ-WWKL. Hence ε-WWKL $\not\leq_{\mathrm{W}}$ δ-WWKL □

The separation is purely topological, i.e., Weihrauch reducibility can be replaced by its continuous counterpart.

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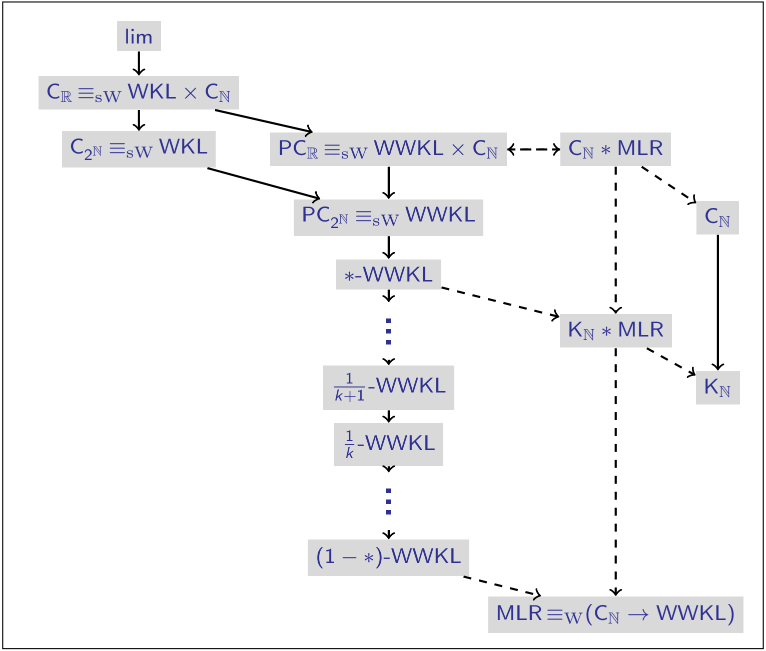
Weak Weak Kőnig's Lemma - The Uniform Scenario



Compositional Product and Implication

The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

Definition

For two mathematical problem f, g we define

- $f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f,\ g_0 \leq_{\mathrm{W}} g\}$ compos. product
- $g \to f := \min\{h : f \leq_{\mathrm{W}} g * h\}$ implication

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$\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$.

Proof. $(\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL}) \leq_{\mathrm{W}} \mathsf{MLR}$: It suffices to prove $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{MLR}$, which follows from Kučera's Lemma.

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Proposition (B., Gherardi and Hölzl 2015)

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Proof. This is a consequence of van Lambalgen's Theorem. □

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The class of functions $f \leq_{\mathrm{W}} \mathsf{MLR}$ is closed under composition.

Jumps

Definition

The jump $f' :\subseteq X \rightrightarrows Y$ of $f :\subseteq X \rightrightarrows Y$ is the same problem, but with the input representation δ of X replaced by $\delta' := \delta \circ \lim$.

A name of an object $x \in X$ with respect to δ' is a sequence that converges to a name with respect to δ. Examples:

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Jump Inversion

Theorem (B., Hölzl and Kuyper 2016)

1. $f' \leq_{\mathrm{W}} g'$ *relative to* $p \Longrightarrow f \leq_{\mathrm{W}} g$ *relative to* p'.
2. $f' \leq_{\mathrm{sW}} g'$ *relative to* $p \Longrightarrow f \leq_{\mathrm{sW}} g$ *relative to* p'.

Proof. Jump Control Theorem (B., Hendtlass and Kreuzer 2015):

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{\;F\;} & \mathbb{N}^{\mathbb{N}} \\ {\scriptstyle \lim}\downarrow & & \downarrow{\scriptstyle \lim} \\ \mathbb{N}^{\mathbb{N}} & \xrightarrow[\;G\;]{} & \mathbb{N}^{\mathbb{N}} \end{array}$$

If there exist a continuous F such that the diagram commutes, then G is continuous. □

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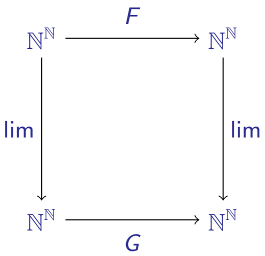

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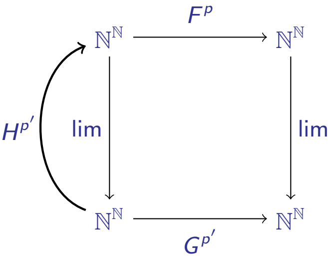
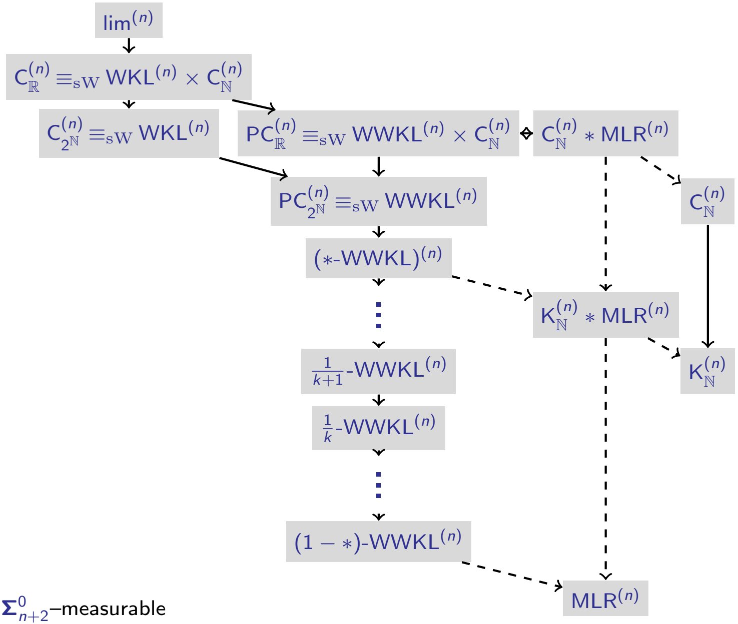


If there exist F computable relative to p such that the diagram commutes, then G is computable relative to p'. □

Weak Weak Kőnig's Lemma - Jumps (work in progress)



$\mathbf{\Sigma}^0_{n+2}$–measurable

Further Notions of Randomness

Theorem (Hölzl and Miyabe 2015)

$\mathsf{WR} <_{\mathrm{W}} \mathsf{SR} <_{\mathrm{W}} \mathsf{CR} <_{\mathrm{W}} \mathsf{MLR} <_{\mathrm{W}} \mathsf{W2R} <_{\mathrm{W}} 2\text{-}\mathsf{RAN}$.

Proof. The strictness has been proved using hyperimmune degrees, high degrees and minimal degrees. □

- WR: Kurtz random
- SR: Schnorr random
- CR: computable random
- W2R: weakly 2-random
- n-RAN: n-random

Question

Find other characterizations of randomness notions R of the form $R \equiv_{\mathrm{W}} (A \to B)$, e.g., $1\text{-}\mathsf{GEN} \equiv_{\mathrm{W}} (? \to \mathsf{BCT}_0')$.

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Uniform Theorem of Kurtz

Theorem of Kurtz. Every 2–random computes a 1–generic.

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1-GEN $<_W$ 2-RAN.

Proof. (Idea) We apply the "fireworks technique" of Rumyantsev and Shen to get a uniform reduction. □

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$\mathsf{BCT}_0' \not\leq_W \mathsf{WWKL}^{(n)}$ *for all* $n \in \mathbb{N}$.

Proof. (Idea) There exists a co-c.e. comeager set $A \subseteq 2^{\mathbb{N}}$ such that no point of A is low for Ω. $\mathsf{WWKL}^{(n)}$ has a realizer that maps computable inputs to outputs that are low for Ω for $n \geq 1$. □

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- A point $x \in \mathbb{R}$ is captured by a sequence $\mathcal{I} = (I_n)_n$ of open intervals, if for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ with $\mathrm{diam}(I_n) < \varepsilon$ and $x \in I_n$.
- $\mathcal{I}$ is a Vitali cover of $A \subseteq \mathbb{R}$, if every $x \in A$ is captured by $\mathcal{I}$.
- $\mathcal{I}$ eliminates A, if the I_n are pairwise disjoint and $\lambda(A \setminus \bigcup \mathcal{I}) = 0$ (where λ denotes the Lebesgue measure).

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Over RCA_0 *the Vitali Covering Theorem is equivalent to Weak Weak Kőnig's Lemma* WWKL_0.

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- $\mathcal{I}$ is called saturated, if $\mathcal{I}$ is a Vitali cover of $\bigcup \mathcal{I} = \bigcup_{n=0}^{\infty} I_n$.

Definition (Contrapositive versions of the Vitali Covering Theorem)

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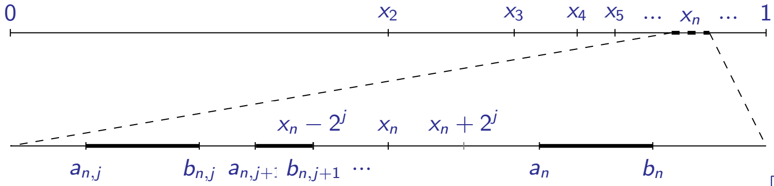
Vitali Covering Theorem

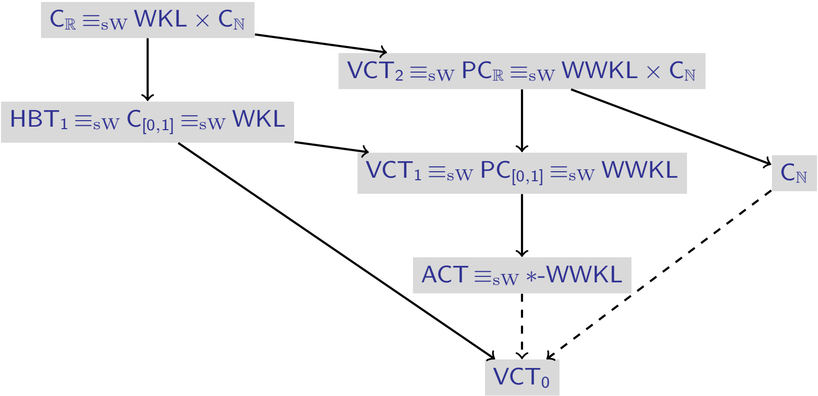
Proof.

- The proof of computability of VCT_0 is based on a construction that repeats steps of the classical proof of the Vitali Covering Theorem (and is not just based on a waiting strategy).
- The proof of $\mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{WWKL}$ is based on the equivalence chain $\mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{PC}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WWKL}$.
- We use a Lemma by Brown, Giusto and Simpson on "almost Vitali covers" in order to prove $\mathsf{VCT}_2 \leq_{\mathrm{sW}} \mathsf{WWKL} \times \mathsf{C}_{\mathbb{N}}$. The harder direction is the opposite one for which it suffices to show $\mathsf{C}_{\mathbb{N}} \times \mathsf{VCT}_2 \leq_{\mathrm{sW}} \mathsf{VCT}_2$ by an explicit construction:



□

Vitali Covering Theorem in the Weihrauch Lattice



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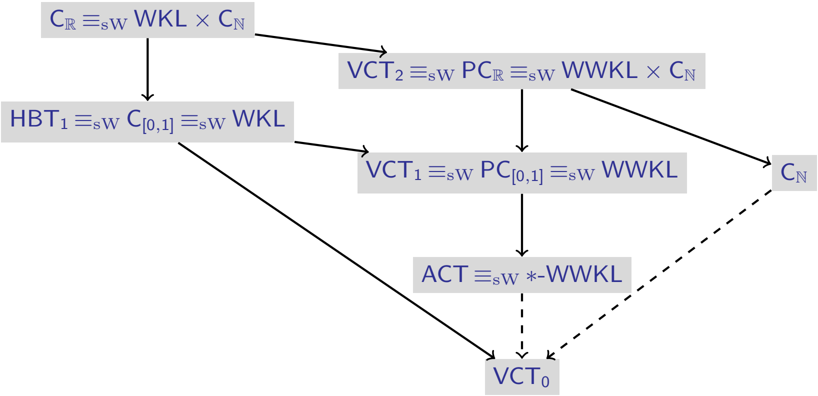

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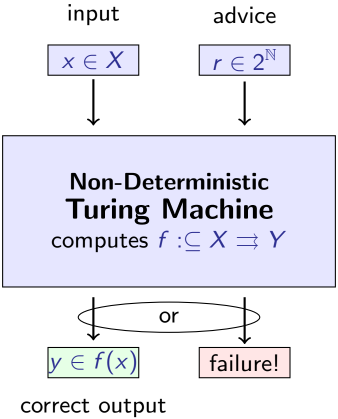
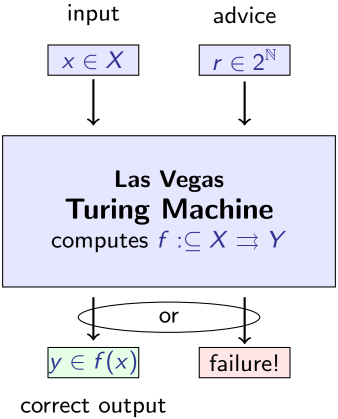
Las Vegas and Monte Carlo Computability

Non-Deterministic Turing Machines



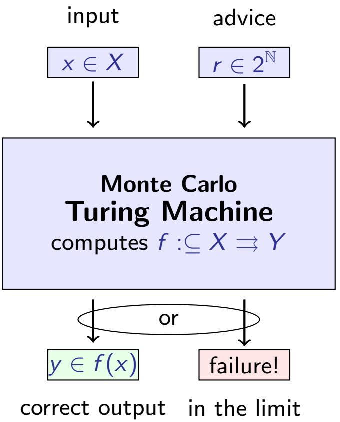
Condition: $(\forall x \in \mathrm{dom}(f))\ \{r \in R : r$ does not fail with $x\} \neq \emptyset$

Las Vegas Turing Machines



Condition: $(\forall x \in \mathrm{dom}(f))\ \mu\{r \in R : r \text{ does not fail with } x\} > 0$

Monte Carlo Turing Machines



Condition: $(\forall x \in \mathrm{dom}(f))\ \mu\{r \in R : r \text{ does not fail with } x\} > 0$

Non-Deterministic Computability

Proposition (B., de Brecht and Pauly 2012)

$f \leq_W \mathrm{WKL} \iff f$ *is non-deterministically computable.*

Non-deterministically computable functions (in this model) were first introduced and studied by Martin Ziegler.

Theorem (Gherardi and Marcone 2009)

The class of $f \leq_W \mathrm{WKL}$ *is closed under composition.*

There are at least three independent proofs:

- The original proof in terms of the separation problem.
- A proof by B. and Gherardi in terms of Kleene's ternary logic.
- A very simple proof in terms of non-deterministically computable functions by B., de Brecht and Pauly.

Corollary

$\mathrm{WKL} \equiv_W \mathrm{WKL} * \mathrm{WKL}$.

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Las Vegas Computability

Proposition (B., Gherardi and Hölzl 2015)

$f \leq_{\mathrm{W}} \mathrm{WWKL} \iff f$ *is Las Vegas computable.*

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Can be proved as for WKL in terms of Las Vegas computable functions with an additional application of Fubini's Theorem.

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Las Vegas computable functions are closed under composition.

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Monte Carlo Computability

Proposition (B., Hölzl and Kuyper 2016)

$f \leq_W \mathrm{PC}'_{\mathbb{R}} \equiv_W \mathrm{WWKL}' \times \mathrm{C}'_{\mathbb{N}} \iff f$ *is Monte Carlo computable.*

This result is based on a classification of positive G_δ–choice by B., Hölzl, Nobrega and Pauly.

Theorem (Bienvenu and Kuyper 2016)

$\mathrm{WWKL}' * \mathrm{WWKL}' \equiv_W \mathrm{PC}'_{2^{\mathbb{N}}} * \mathrm{PC}'_{2^{\mathbb{N}}} \equiv_W \mathrm{PC}'_{\mathbb{R}} * \mathrm{PC}'_{\mathbb{R}} \equiv_W \mathrm{PC}'_{\mathbb{R}}$.

This contrasts $\mathrm{WKL}' * \mathrm{WKL}' \equiv_W \mathrm{WKL}''$.

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Monte Carlo computable functions are closed under composition.

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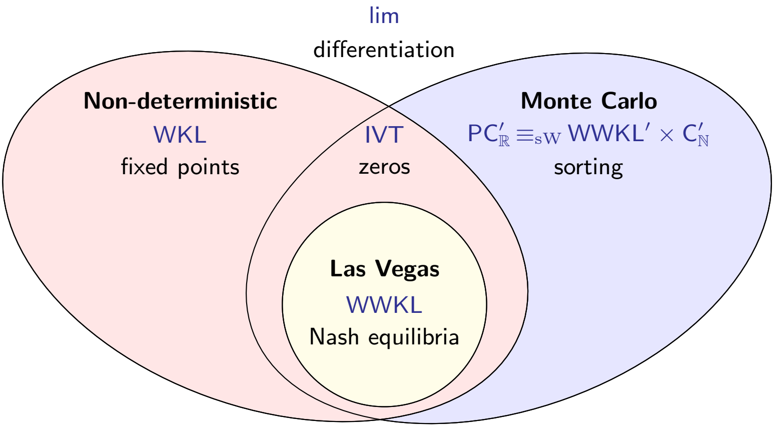
Classes of Computability



Sorting

Definition

$\text{SORT}_n : \{0,1,\ldots,n-1\}^{\mathbb{N}} \to \{0,1,\ldots,n-1\}^{\mathbb{N}}$ is defined by

$$\text{SORT}_n(p) := 0^{k_0} 1^{k_1} \ldots (m-1)^{k_{m-1}} \widehat{m}$$

if $m < n$ is the smallest digit that appears infinitely often in p and each digit $i < m$ appears exactly k_i times in p.

0	3	2	1	3	1	2	1	3	4	3	4	3	4	3	⋯

↓ SORT_5

0	1	1	1	2	2	3	3	3	3	3	3	3	3	3	⋯

Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)

- $\text{C}_{\mathbb{N}} \leq_{\text{sW}} \text{SORT}_2 \leq_{\text{sW}} \text{C}'_{\mathbb{N}}$
- $\text{IVT} \leq_{\text{W}} \text{SORT}_2 \leq_{\text{W}} \text{WWKL}'$

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0	3	2	1	3	1	2	1	3	4	3	4	3	4	3	⋯

$\downarrow \mathrm{SORT}_5$

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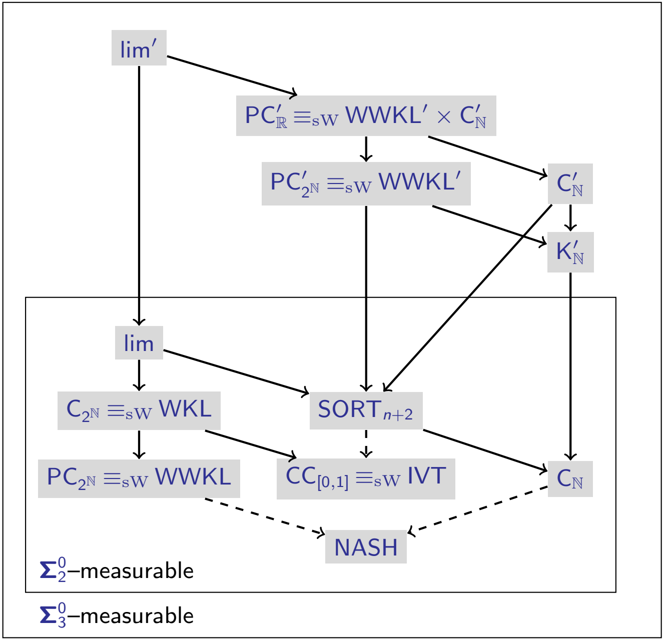
Sorting in the Weihrauch Lattice



Sorting and Algebraic Machine Models

Besides COH sorting is the only problem that we know that is low_2 but not low in the following sense.

Proposition (Neumann and Pauly, B., Hölzl and Kuyper 2016)

$\lim * \lim * \text{SORT}_2 \leq_W \lim * \lim$ *and* $\lim * \text{SORT}_2 \not\leq_W \lim$.

Neumann and Pauly proved that SORT_2^* characterizes the class of functions computable by certain algebraic machine models.

Corollary

BSS computable functions $f : \mathbb{R}^* \to \mathbb{R}^*$ *are computable on Monte Carlo machines.*

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References

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