

Schnorr randomness for noncomputable measures

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Randomness: A revisionist history

Computable/constructive analysis by example

Classical analysis

If $p > 1$, then the series $\sum n^{-p}$ converges.

Computable analysis (computable version)

If $p > 1$ **and p is computable**, then the series $\sum n^{-p}$ converges **with a computable rate of convergence**.

Computable analysis (relativized version)

If $p > 1$, then the series $\sum n^{-p}$ converges **with a rate of convergence uniformly computable from (a name for) p** .

Constructive analysis

If $p > 1$, then the series $\sum n^{-p}$ converges.

Computable/constructive analysis

Computable/constructive definitions

- Computable real numbers
- Computable Polish spaces
- Computable continuous functions
- Computable lower semicontinuous functions
- Computable integrable functions
- Computable probability measures
- Etc.

“Constructive null sets”

Martin-Löf 1966 (Emphasis mine)

*In this paper it is shown that the random elements as defined by Kolmogorov possess all conceivable statistical properties of randomness. They can equivalently be considered as the elements which withstand a certain universal stochasticity test. The definition is extended to infinite binary sequences and it is shown that the non random sequences form a **maximal constructive null set**.*

Schnorr 1969 (Emphasis mine)

*Martin-Löf has defined random sequences to be those sequences which withstand a certain universal stochasticity test. On the other hand one can define a sequence to be random if it is not contained in any **[set] of measure zero in the sense of Brouwer**. Both definitions imply that these random sequences possess all statistical properties which can be checked by algorithms. We draw a comparison between the two concepts of **constructive null sets** and prove that they induce concepts of randomness which are not equivalent. The union of all **[sets] of measure zero in the sense of Brouwer** is a proper subset of the **universal constructive null set** defined by Martin-Löf.*

Martin-Löf and Schnorr randomness

Let μ be a computable measure on $2^{\mathbb{N}}$.

Definition

A **Martin-Löf sequential μ -test** is a sequence $(U^n)_{n \in \mathbb{N}}$ where

- $U^n \subseteq 2^{\mathbb{N}}$ is Σ_1^0 in n , and
- $\mu(U^n) \leq 2^{-n}$.

A sequence $x \in 2^{\mathbb{N}}$ is **Martin-Löf μ -random** if $x \notin \bigcap_n U^n$ for all Martin-Löf sequential μ -tests $(U^n)_{n \in \mathbb{N}}$.

Definition

A **Schnorr sequential μ -test** is a sequence $(U^n)_{n \in \mathbb{N}}$ where

- $U^n \subseteq 2^{\mathbb{N}}$ is Σ_1^0 in n ,
- $\mu(U^n) \leq 2^{-n}$, and
- $n \mapsto \mu(U^n)$ is computable.

A sequence $x \in 2^{\mathbb{N}}$ is **Schnorr μ -random** if $x \notin \bigcap_n U^n$ for all Schnorr sequential μ -tests $(U^n)_{n \in \mathbb{N}}$.

A.E. theorems and Schnorr randomness

Lebesgue Differentiation Theorem (Pathak–Rojas–Simpson; R.)

Assume $f: [0,1] \rightarrow \mathbb{R}$ is a computably integrable function and x is Schnorr random, then $\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(x) dx$ converges to $f(x)$.

Carleson's Theorem (Franklin–McNicholl–R.)

Assume $f: [0,2\pi] \rightarrow \mathbb{C}$ is a computably square integrable function and x is Schnorr random, then the Fourier series of f at x converges to $f(x)$.

Ergodic theorem for ergodic measures (Gács–Hoyrup–Rojas)

Assume (X, μ, T) is a computable ergodic measure-preserving system and x is Schnorr μ -random, then $\frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$ converges to $\int f d\mu$.

Constructive a.e. theorems

If an a.e. theorem is constructively provable¹ then there is a computable version of the theorem holding for Schnorr randomness.

¹In, say, the frameworks of Brouwer, Bishop, or Demuth.

Randomness for noncomputable measures

- One of the most important developments in Martin-Löf randomness was a theory of randomness for **noncomputable measures**.
- Martin-Löf randomness for noncomputable measures
 - Martin-Löf (1966); Levin (1976); Gács (2005); Takahashi (2005); Takahashi (2008); Reimann (2008); Hoyrup–Rojas (2009); Kjos-Hanssen (2009); Bienvenu–Gács–Hoyrup–Rojas–Shen (2011); Hoyrup (2012); Kjos-Hanssen (2012); Diamondstone–Kjos-Hanssen (2012); Bienvenu–Monin (2012); Day–Miller (2013); Day–Reimann (2014); Reimann–Slaman (2015); Miller–R. (201x); and many more
- Schnorr randomness for noncomputable measures
 - Schnorr's Book (1971);
 - Schnorr–Fuchs (1977); and ...
 - **Jason Rute. Schnorr randomness for noncomputable measures. Submitted. Available on arXiv.**

The goals of this talk

- Give the **right definition** of Schnorr randomness for noncomputable measures
- Develop the theory of Schnorr randomness for noncomputable measures
- Apply this to solve open questions about Schnorr randomness for **computable measures**
- Apply this to deep questions about randomness **as a general concept**

Schnorr randomness for noncomputable measures

Build your own definition...

Definition (for a computable measure μ)

A **Schnorr sequential μ -test** is a sequence $(U^n)_{n \in \mathbb{N}}$ where

- $U^n \subseteq 2^{\mathbb{N}}$ is Σ_1^0 in n ,
- $\mu(U^n) \leq 2^{-n}$, and
- $n \mapsto \mu(U^n)$ is computable.

A sequence $x \in 2^{\mathbb{N}}$ is **Schnorr μ -random** if $x \notin \bigcap_n U^n$ for all Schnorr sequential μ -tests $(U^n)_{n \in \mathbb{N}}$.

What would you do?

What is your definition of Schnorr randomness for noncomputable measures?

Schnorr randomness for noncomputable measures

My definition

Integral tests for computable measures

Let μ be a computable measure.

Definition (Levin)

A **ML integral μ -test** is a lower semicomputable function $t: 2^{\mathbb{N}} \rightarrow [0, \infty]$ such that $\int t d\mu \leq 1$.

Theorem (Levin?)

x is Martin-Löf μ -random iff $t(x) < \infty$ for all Martin-Löf integral μ -tests t .

Definition (Miyabe)

A **Schnorr integral μ -test** is a lower semicomputable function $t: 2^{\mathbb{N}} \rightarrow [0, \infty]$ such that $\int t d\mu \leq 1$ and $\int t d\mu$ is computable.

Theorem (Miyabe)

x is Schnorr μ -random iff $t(x) < \infty$ for all Schnorr integral μ -tests t .

Randomness for noncomputable measures

Definition (Gács, following Levin)

- A **uniform ML integral test** is a function t such that
 - $t: \text{Prob}(2^{\mathbb{N}}) \times 2^{\mathbb{N}} \rightarrow [0, \infty]$ is lower semicomputable, and
 - $\int t(\mu, x) d\mu(x) \leq 1$ for all μ
- $x_0 \in \text{ML}_{\mu_0}$ iff $t(\mu_0, x_0) < \infty$ for all uniform ML integral tests t .

Definition (R.)

- A **uniform Schnorr integral test** is a function t such that
 - $t: \text{Prob}(2^{\mathbb{N}}) \times 2^{\mathbb{N}} \rightarrow [0, \infty]$ is lower semicomputable
 - $\int t(\mu, x) d\mu(x) \leq 1$ for all μ , and
 - $\mu \mapsto \int t(\mu, x) d\mu$ is computable.
- $x_0 \in \text{SR}_{\mu_0}$ iff $t(\mu_0, x_0) < \infty$ for all unif. Schnorr integral tests t .

Randomness for noncomputable oracles and measures

Let \mathbb{X} and \mathbb{A} be computable metric spaces (e.g. \mathbb{R} , $C(2^{\mathbb{N}})$, $L^1(\mathbb{X}, \mu)$, $\text{Prob}(2^{\mathbb{N}})$).

Definition (R.)

- A **uniform Schnorr integral test** is a function t such that
 - $t: \mathbb{A} \times \text{Prob}(\mathbb{X}) \times \mathbb{X} \rightarrow [0, \infty]$ is lower semicomputable
 - $\int t(a, \mu, x) d\mu(x) \leq 1$ for all $a \in \mathbb{A}$ and $\mu \in \text{Prob}(\mathbb{X})$, and
 - $a, \mu \mapsto \int t(a, \mu, x) d\mu$ is computable.
- $x_0 \in \text{SR}_{\mu_0}^a$ iff $t(a_0, \mu_0, x_0) < \infty$ for all unif. Schnorr integral tests t .

Notation: SR_{μ}^a

- The oracle a is in the superscript.
- Measure μ is in the subscript.
- The relativization is always uniform unless stated otherwise.

Schnorr randomness for noncomputable measures

Justification

New definition agrees on computable measures

Theorem

If μ_0 is a computable measure, the following are equivalent:

- x_0 is Schnorr μ_0 -random (original definition)
- $x_0 \in \mathbf{SR}_{\mu_0}$ (extended definition in this talk)

Proof Sketch.

(\Rightarrow) If $x \notin \mathbf{SR}_{\mu_0}$, there is a Sch. uniform integral test t such that $t(\mu_0, x_0) = \infty$.
Then $t(x_0) := t(\mu_0, x_0)$ is a Schnorr integral test and x_0 is not Schnorr μ_0 -random.

(\Leftarrow) If x_0 is not μ_0 -Schnorr rand., there is a Sch. integral μ_0 -test t s.t. $t(x_0) = \infty$.
We must extend t to a uniform integral test $t(\mu, x)$.
Find computable $f: \mathbb{N} \times \mathbb{X} \rightarrow [0, \infty)$ such that $t(x) = \sum_n f(n, x)$.

Let $t(\mu, x) := \sum_n f(n, x) \cdot \frac{\int f(n, x) d\mu_0(x)}{\int f(n, x) d\mu}$. This works. □

Van Lambalgen for noncomputable measures

Let μ and ν be (noncomputable) measures (on \mathbb{X} and \mathbb{Y}).

Recall, $\mu \otimes \nu$ is the measure given by

$$(\mu \otimes \nu)(A \times B) = \mu(A) \cdot \nu(B) \quad (A \subseteq \mathbb{X}, B \subseteq \mathbb{Y}).$$

Theorem (Folklore, following Van Lambalgen [computable case])

$$(x, y) \in \text{ML}_{\mu \otimes \nu} \iff x \in \text{ML}_{\mu}^{\nu} \wedge y \in \text{ML}_{\nu}^{x, \mu}.$$

Theorem (R., following Miyabe–R. [computable case])

$$(x, y) \in \text{SR}_{\mu \otimes \nu} \iff x \in \text{SR}_{\mu}^{\nu} \wedge y \in \text{SR}_{\nu}^{x, \mu}.$$

Schnorr randomness for noncomputable measures

Alternate definitions (not equivalent to mine)

Schnorr–Fuchs definition

Definition (Schnorr–Fuchs (1977))

$x \in 2^{\mathbb{N}}$ is Schnorr–Fuchs μ -random if there is no test of this form:

- a computable measure ν on $2^{\mathbb{N}}$ and
- a computable order function f such that

$$\exists^{\infty} n \quad \frac{\nu(x \upharpoonright n)}{\mu(x \upharpoonright n)} \geq f(n).$$

Observation

This is a “blind randomness” notion which doesn’t use the computability of the measure μ . In blind randomness, measures which are “locally similar”, but not necessarily “computably similar,” still have the same randoms.

Theorem (R.)

The Schnorr–Fuchs definition is different from the one in this talk. (Although, it likely agrees for nice measures—e.g. Bernoulli measures.)

Nonuniform definition

Definition

A **nonuniform Schnorr sequential μ -test** is a sequence $(U^n)_{n \in \mathbb{N}}$ where

- $U^n \subseteq 2^{\mathbb{N}}$ is Σ_1^0 in μ and n ,
- $\mu(U^n) \leq 2^{-n}$, and
- $n \mapsto \mu(U^n)$ is μ -computable.

A point x is **nonuniformly Schnorr μ -random** if $x \notin \bigcap_n U^n$ for all nonuniform Schnorr sequential μ -tests $(U^n)_{n \in \mathbb{N}}$.

Theorem (R.)

- The nonuniform definition is strictly stronger than the uniform definition.
- The two definitions differ even on Bernoulli measures [unpublished].
- The nonuniform definition does not satisfy Van Lambalgen's theorem.
- (It does not matter if we use non-uniform integral tests.)

Uniform Schnorr sequential tests

Definition

A **uniform Schnorr sequential test** is a family $\{U_\mu^n\}_{n \in \mathbb{N}, \mu \in \text{Prob}(\mathbb{X})}$ where

- $U_\mu^n \subseteq 2^{\mathbb{N}}$ is Σ_1^0 in μ and n ,
- $\mu(U_\mu^n) \leq 2^{-n}$ for all μ , and
- $\mu, n \mapsto \mu(U_\mu^n)$ is computable.

A point x is **“Schnorr μ_0 -random”** if $x \notin \bigcap_n U_{\mu_0}^n$ for all uniform Schnorr sequential tests $\{U_\mu^n\}$.

Theorem (Hoyrup [personal communication])

For any space, the only uniform Schnorr sequential test is the trivial one:

$$U_\mu^n = \emptyset \quad (\text{for all } n \text{ and } \mu)!$$

With this definition, all points would be “Schnorr μ -random.”

Schnorr randomness for noncomputable measures

Other equivalent definitions

Uniform sequential tests for “nice” classes of measures

(Re)definition

A **uniform Schnorr sequential test** is a family $\{U_\mu^n \mid n \in \mathbb{N}, \mu \in K\}$ for some Π_1^0 set $K \subseteq \text{Prob}(\mathbb{X})$ such that

- $U_\mu^n \subseteq 2^{\mathbb{N}}$ is Σ_1^0 in μ and n ,
- $\mu(U_\mu^n) \leq 2^{-n}$ for all $\mu \in K$, and
- $\mu, n \mapsto \mu(U_\mu^n)$ is computable (for $\mu \in K$).

Proposition (R. [unpublished])

Let Ber be the class of Bernoulli measures. TFAE for $\beta_0 \in \text{Ber}$.

- $x \in \text{SR}_{\beta_0}$
- $x \notin \bigcap_n U_{\beta_0}^n$ for all uniform Schnorr sequential tests $\{U_\beta^n \mid n \in \mathbb{N}, \beta \in \text{Ber}\}$.

- Schnorr's 1971 books gives a uniform sequential test definition of Schnorr randomness for noncomputable Bernoulli measures.
- His definition is equivalent to the one in this talk [unpublished].

Randomness with respect to a name

Theorem (R.)

The following are equivalent:

- $x \in \text{SR}_\mu$
- $x \in \text{SR}_\mu^r$ for some Cauchy name $r \in \mathbb{N}^{\mathbb{N}}$ for μ

Theorem (R.)

For a measure $\mu_0 \in \text{Prob}(\mathbb{X})$ with a Cauchy name $r_0 \in \mathbb{N}^{\mathbb{N}}$, TFAE:

- $x \in \text{SR}_{\mu_0}^{r_0}$
 - $x \notin \bigcap_n U_{\mu_0}^{n; r_0}$ for all uniform Schnorr sequential tests
- $$\left\{ U_{\mu}^{n; r} \mid n \in \mathbb{N}, (r, \mu) \in \mathbb{N}^{\mathbb{N}} \times \text{Prob}(\mathbb{X}), r \text{ is a Cauchy name for } \mu \right\}.$$

- These results allow us to relativize most (if not all) results concerning Schnorr randomness and computable analysis.

Disintegration theorems

Exchangeable measures

Exchangeable measures

A measure μ on $2^{\mathbb{N}}$ is **exchangeable** if it is preserved under rearrangement of bits. For example, if μ is exchangeable, then

$$\mu \{x(1)x(0)x(3)x(2)\dots \mid x \in A\} = \mu(A)$$

for all measurable sets A .

de Finetti's Theorem

A measure is exchangeable iff it is a **mixture of Bernoulli measures**, i.e. there is a measure ξ , on the set of Bernoulli measures, such that

$$\mu(E) = \int \beta(E) d\xi(\beta).$$

Schnorr randomness and exchangeable measures

$$\mu(E) = \int \beta(E) d\xi(\beta)$$

Theorem (Freer–Roy)

An exchangeable measure μ is computable iff the corresponding disintegration measure ξ is computable.

Theorem (R. [unpublished])

If μ is a computable exchangeable measure then the following are equivalent:

- $x \in SR_\mu$
- $x \in SR_\beta$ for some Bernoulli measure $\beta \in SR_\xi$.

Remark [unpublished]

This holds for other disintegration theorems, e.g. the ergodic decomposition.

Van Lambalgen for Kernels

Let μ be a (noncomputable) measure on \mathbb{X} .

Let $\kappa : \mathbb{X} \rightarrow \text{Prob}(\mathbb{Y})$ be a (noncomputable) kernel.

The measure κ_x represents the measure on $\{x\} \times \mathbb{Y}$.

Define

$$(\mu * \kappa)(A \times B) = \int_A \kappa_x(B) d\mu(x) \quad (A \subseteq \mathbb{X}, B \subseteq \mathbb{Y}).$$

Theorem (Takahashi)

$$(x, y) \in \text{ML}_{\mu * \kappa}^{\kappa} \quad \Leftrightarrow \quad x \in \text{ML}_{\mu}^{\kappa} \wedge y \in \text{ML}_{\kappa(x)}^{x, \mu, \kappa}.$$

Theorem (R. [half published])

$$(x, y) \in \text{SR}_{\mu * \kappa}^{\kappa} \quad \Leftrightarrow \quad x \in \text{SR}_{\mu}^{\kappa} \wedge y \in \text{SR}_{\kappa(x)}^{x, \mu, \kappa}.$$

Van Lambalgen for maps

- Let μ be a measure on \mathbb{X} and let ν be a measure on \mathbb{Y} .
- Let $T: (\mathbb{X}, \mu) \rightarrow (\mathbb{Y}, \nu)$ be a measurable-preserving map.
- Let $y \mapsto \mu(\cdot | T = y)$ be the **conditional probability of T** .
- If $\mu(\{x | T(x) = y\}) > 0$, then

$$\mu(A | T = y) = \frac{\mu(A \cap \{x | T(x) = y\})}{\mu\{x | T(x) = y\}} \quad (A \subseteq \mathbb{X}).$$

- More generally, $y \mapsto \mu(\cdot | T = y)$ is a map $\mathbb{Y} \rightarrow \text{Prob}(\mathbb{X})$ defined via

$$\mu(A \cap T^{-1}(B)) = \int_B \mu(A | T = y) d\nu(y) \quad (A \subseteq \mathbb{X}, B \subseteq \mathbb{Y}).$$

Theorem (R.)

If μ, ν, T , and $y \mapsto \mu(\cdot | T = y)$ are computable,

$$(x \in \text{SR}_\mu \wedge y = T(x)) \Leftrightarrow (y \in \text{SR}_\nu \wedge x \in \text{SR}_{\mu(\cdot | T=y)}^y)$$

- Also holds for ML randomness.

Randomness conservation results

Theorem (R.)

If μ , ν , T , and $y \mapsto \mu(\cdot \mid T = y)$ are computable,

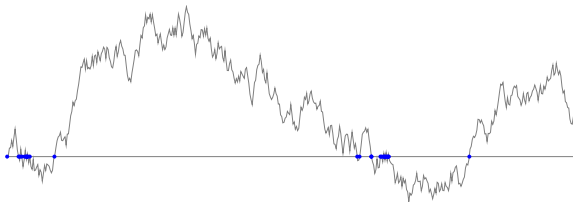
$$\left(x \in \text{SR}_\mu \wedge y = T(x) \right) \Leftrightarrow \left(y \in \text{SR}_\nu \wedge x \in \text{SR}_{\mu(\cdot \mid T=y)}^y \right)$$

Corollary (R.)

If μ , ν , T , and $y \mapsto \mu(\cdot \mid T = y)$ are computable:

- 1 If $x \in \text{SR}_\mu$, then $T(x) \in \text{SR}_\nu$. (Randomness conservation)
- 2 If $y \in \text{SR}_\nu$, then $T(x) = y$ for some $x \in \text{SR}_\mu$. (No randomness from nothing)

Solving a question about SR Brownian motion



Question (Bienvenu; and others)

- 1 If B is a Schnorr random Brownian motion, is $B(1)$ also Schnorr random (for the Lebesgue measure)?
- 2 If x is Schnorr random (for the Lebesgue measure), does $B(1) = x$ for some Schnorr random Brownian motion B ?

Theorem (R.)

Yes!

Schnorr random Brownian motion

Being Schnorr random for the Gaussian measure (normal distribution) and the Lebesgue measure are known to be equivalent.

Theorem (R.)

- 1 If B is a SR Brownian motion, then $B(1)$ is Gaussian SR.
- 2 If x is Gaussian SR, then $B(1) = x$ for some SR Brownian motion.

Proof.

The map $B \mapsto B(1)$ is

- a computable map of type $C([0, 1]) \rightarrow \mathbb{R}$.
- is a measure preserving map between the Wiener measure P (the measure of Brownian motion) and the Gaussian measure N .
- has a computable conditional probability map, namely $P(\cdot \mid B(1) = y)$ is the probability distribution of a Brownian bridge landing at y .

(1) follows from randomness preservation.

(2) follows from no-randomness from nothing. □

Future Projects

A new reducibility

Theorem (Based on personal communication with Miller; attributed to Nies)

The following are equivalent for oracles $a \in \mathbb{X}$ and $b \in \mathbb{Y}$.

- 1 $ML_{\mu}^a \subseteq ML_{\mu}^b$ for all (non-computable) measures μ
- 2 a computes b (e.g. $a \geq_T b$)

Question

The following are equivalent for oracles $a \in \mathbb{X}$ and $b \in \mathbb{Y}$.

- 1 $SR_{\mu}^a \subseteq SR_{\mu}^b$ for all (non-computable) measures μ
- 2 ???

Partial Answer / Conjecture

The following reducibility is sufficient and conjectured to be equivalent:

- $a \geq b$ ($a \in \mathbb{X}$ and $b \in \mathbb{Y}$) if there is a Π_1^0 set $a \in K \subseteq \mathbb{X}$ and a computable map $f: K \rightarrow \mathbb{Y}$ such that $f(a) = b$. (For $\mathbb{X} = \mathbb{Y} = 2^{\mathbb{N}}$, this is tt-reducibility.)

The more-random-than relation

Goal

Come up with a natural “more-random-than” relation which tells us that one random (x, μ) is more random than another (y, ν) .

Candidate reducibilities

\leq_{LR} , \leq_{Sch} , \leq_K , and \leq_{vL} and many more. (See Chapter 10 of Downey and Hirschfeldt.)

A new approach (informally)

View of randoms as “infinitesimally small point-masses” and compare their masses.

Tools

This approach uses conditional probabilities, Schnorr randomness for noncomputable measures, and Van Lambalgen’s theorem for maps.

Comparing amount of randomness

A new approach (the rough idea)

Assume $y = T(x)$ for a computable measure preserving map $T: (\mathbb{X}, \mu) \rightarrow (\mathbb{Y}, \nu)$ with a computable conditional probability. Then

$$\mu(x) = \mu(\{x\} | T = y) \cdot \nu(y)$$

Define

$$\frac{\mu(dx)}{\nu(dy)} = \mu(\{x\} | T = y).$$

- I can show $\mu(dx)/\nu(dy)$ is consistent for $x \in \text{SR}_\mu$ and $y \in \text{SR}_\nu$.
- I can also show the following:

Theorem (R. [unpublished])

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is computable in $C^1(\mathbb{R})$, $x \in \mathbb{R}$ is Schnorr random for the Lebesgue measure λ , and $f'(x) \neq 0$, then $f(x)$ is λ -Schnorr random and

$$\frac{\lambda(df(x))}{\lambda(dx)} = |f'(x)|.$$

Lots of work to do

Papers on ML randomness for non-computable measures

Martin-Löf (1966); Levin (1976); Gács (2005); Takahashi (2005); Takahashi (2008); Reimann (2008); Hoyrup-Rojas (2009); Kjos-Hanssen (2009); Bienvenu-Gacs-Hoyrup-Rojas-Shen (2011); Hoyrup (2012); Kjos-Hanssen (2012); Diamondstone-Kjos-Hanssen (2012); Bienvenu-Monin (2012); Day-Miller (2013); Day-Reimann (2014); Reimann-Slaman (2015); Miller-R. (201x); and many more

Goal

Transfer these results to Schnorr randomness

Conclusion

In summary

In this talk we ...

- We gave a novel definition of Schnorr randomness for noncomputable measures.
- We argued for its correctness and compared it to other possible definitions.
- We showed applications to disintegration theorems.
- We showed applications to Schnorr randomness for Brownian motion.
- We outlined how these results fit into a larger research program.

Thank You!

For more details, see my paper:

- Jason Rute. Schnorr randomness for noncomputable measures. Submitted. [arXiv:1607.04679](https://arxiv.org/abs/1607.04679)

These slides will be available on my webpage:

<http://www.personal.psu.edu/jmr71/>

Or just Google™ me, “Jason Rute”.

P.S. I am on the job market.