Computability and the Denjoy Hierarchy

Linda Brown Westrick
University of Connecticut

December 9, 2016
BIRS/CMO Oaxaca

Lebesgue integration

A function $F : [0,1] \to \mathbb{R}$ is called **absolutely continuous** if for every ε there is a δ such that whenever $(a_0, b_0), \ldots, (a_k, b_k) \subset [0,1]$ are disjoint intervals satisfying

$$\sum_{i \leq k} b_i - a_i < \delta,$$

then

$$\sum_{i \leq k} |F(b_i) - F(a_i)| < \varepsilon.$$

Theorem (Lebesgue)

The following are equivalent.

1. F is absolutely continuous.
2. There is a Lebesgue integrable function f such that $F(x) = F(0) + \int_0^x f$.
3. F is a.e. differentiable and $f = F'$ is as above.

Lebesgue integration

Limitation: a function can be everywhere differentiable without being absolutely continuous. For example, $x^2 \sin\left(\frac{1}{x^2}\right)$.

Really, this function's derivative should be integrable.

The problem of "recovering the primitive" of an everywhere differentiable function was solved by Denjoy in 1912, via the transfinite process of (narrow) **Denjoy integration**.

Denjoy integration

Theme: Lebesgue integrate and sum up...and if you don't succeed, try again.

Goal: Given f, identify F with $F' = f$ by finding all $F(y) - F(x)$.

Denjoy integration process

- If f is Lebesgue integrable on an open set U containing x, y, set $F(y) - F(x) = \int_x^y f$.
- Let $P^1 = [0,1] \setminus \cup\{U : f \restriction U \text{ is Lebesgue integrable}\}$.
- Given closed $P^\alpha \neq \emptyset$:
  - For (c, d) contiguous to P^α, set $F(d) - F(c)$ and similar by taking limits, if possible. (Impossible $\implies$ f not Denjoy integrable).
  - If $f \restriction P^\alpha \cap U$ is Lebesgue integrable and F is summable on $U \setminus P^\alpha$, for U an open set containing x, y, set

$$F(y) - F(x) = \int_{P^\alpha \cap [x,y]} f + \sum_{(c,d) \in U \setminus P^\alpha} F(d) - F(c).$$

  - $P^{\alpha+1} = P^\alpha \setminus \cup\{U : f \restriction P^\alpha \cap U \text{ is Lebesgue integrable and } F \text{ summable on } U \setminus P^\alpha\}$.
  - At limit stages, intersect all previous P^α.

Technical slide

If U is open and P is closed, F is **summable** on $U \setminus P$ if

$$\sum_{(c,d)\in U\setminus P} \omega(F,[c,d]) < \infty$$

where $\omega(F,[c,d])$ is the oscillation of F on $[c,d]$,
and (c,d) ranges over the connected components of $U \setminus P$.

Descriptive Definition of the Denjoy Integral

Theorem (Denjoy and others)

The following are equivalent.

1. F is $\mathbf{ACG}_*$.
2. There is a **Denjoy** integrable function f such that $F(x) = F(0) + \int_{\mathbf{0}}^{\mathbf{x}} f$.
3. F is a.e. differentiable and $f = F'$ is as above.

Absolutely continuous functions are those obtainable by Lebesgue integration.

ACG_* is the set of functions obtainable by Denjoy integration.

Question

What is the descriptive complexity of ACG_* and its subclasses D_α?

$$D_\alpha := \{F : F \text{ is obtainable by at most } \alpha \text{ steps of Denjoy integration}\}$$

Being absolutely continuous is Π^0_3-complete

Proposition (W.) The set of absolutely continuous functions is Π^0_3-complete.

$D_1 = \{F : F$ is obtainable by at most 1 step of Denjoy integration$\}$
$D_1 =$ is exactly the absolutely continuous functions.

F is absolutely continuous if for every ε there is a δ such that whenever $(a_0, b_0), \ldots, (a_k, b_k) \subset [0, 1]$ are disjoint intervals satisfying $\sum_{i \le k} b_i - a_i < \delta$, then $\sum_{i \le k} |F(b_i) - F(a_i)| < \varepsilon$.

Examples of not absolutely continuous functions:

- Anything with unbounded variation (Π^0_2 failure)
- Cantor's staircase (Π^0_3 failure)

Naively, being absolutely continuous is Π^0_3.
Π^0_3-hardness result produces Cantor staircase-like functions.

The sets D_α for $\alpha > 1$ are $\Sigma_{2\alpha}$

Proposition (W.) The sets D_α for $\alpha > 1$ are $\Sigma_{2\alpha}$.

If $F \in D_\alpha$, the integration process producing F can be recovered in α steps.

Last step: $F \restriction P^{\alpha-1}$ is Lebesgue integrable and F is summable on $[0,1] \setminus P^{\alpha-1}$.

Equivalently: the continuous function $F_{\alpha-1}$ is absolutely continuous.

$$F_{\alpha-1}(x) = \begin{cases} F(x) & \text{if } x \in P^{\alpha-1} \\ \text{interpolate}(x,c,d) & \text{if } x \in (c,d) \text{ contiguous to } P^{\alpha-1}. \end{cases}$$

where interpolate(x,c,d) is a three piece linear function whose maximum and minimum on (c,d) agree with $\max_{x\in[c,d]} F(x)$ and $\min_{x\in[c,d]} F(x)$.

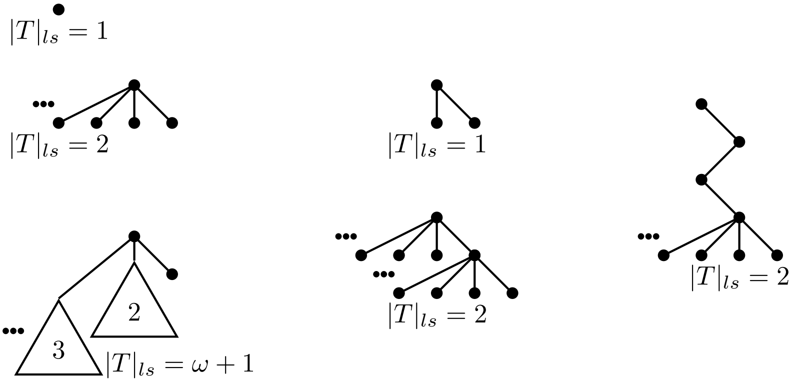
Naively, it could take three jumps to tell if $F_{\alpha-1}$ is absoutely continuous.
But the subtle Cantor's staircase problem, if it ever happens, will happen at F_1, with the integration simulation failing at that level.
So we have only a two-jump problem: detecting unbounded variation.

Limsup rank

Definition

The **limsup rank** of a well-founded tree T is 0 if $T = \emptyset$ and

$$|T|_{ls} = \max([\limsup_n |T_n|_{ls}] + 1, \sup_n |T_n|_{ls}) \text{ otherwise.}$$



Analysis of the Limsup Rank

Let $L_\alpha = \{e : e \text{ codes } T \text{ with } |T|_{ls} \leq \alpha\}$.
The L_α are naively $\Sigma_{2\alpha}$ for $\alpha > 0$.

Theorem (W.)

For each constructive $\alpha > 0$,

$$(\Sigma_{2\alpha}, \Pi_{2\alpha}) \leq_1 (L_\alpha, L_{\alpha+1} \setminus L_\alpha).$$

In other words, if A is $\Sigma_{2\alpha}$-complete, there is recursive f such that for all x,

- $x \in A \to f(x) \in L_\alpha$
- $x \notin A \to f(x) \in L_{\alpha+1} \setminus L_\alpha$

Corollary: For each constructive $\alpha > 1$, $\{e : e \text{ codes } F \in D_\alpha\}$ is $\Sigma_{2\alpha}$-complete.

Proof: Computably transform trees of limsup rank α into functions $F \in D_\alpha$, by piling on bounded variation.

Proposition (W.) ACG_* is Π^1_1-complete.

Definition direction: $F \in ACG_*$ if and only if for every perfect $E \subseteq [0,1]$, there is an interval J such that F is AC_* on the nonempty set $E \cap J$.

F is AC_* on E: like absolute continuity, but the endpoints of the intervals (a_i, b_i) must be in E, and $|F(b_i) - F(a_i)|$ is replaced with $\omega(F, [a_i, b_i])$.

Completeness direction: observe that if T is not WF, then the transformation which takes trees of limsup rank α to $F \in D_\alpha$, now achieves

$$(\Pi^1_1, \Sigma^1_1) \leq_1 (ACG_*, \overline{ACG_*})$$