The Inverse Function Theorems of Lawrence M. Graves

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Hi, here is a question I need your help with.

Let $T$ be tangent to ellipse $E$ at $f$, show that for $p$ in a neighbourhood of $f$,

$$||P_E(p) - P_T(p)|| = o(||p - f||).$$

That is; $P_T$ is the linearisation of $P_E$ at $f$ and 

$$||P_E(p) - P_T(p)||/||p - f|| \to 0, \quad \text{as } p \to f.$$

Since projection onto a line $L$ is linear this will let us show that the D-R operator .....
The theorems

- The Hildebrand-Graves theorem (1927)
- The (Lyusternik-) Graves theorem (1950)
- The Bartle-Graves theorem (1952)

Lawrence Murry Graves (1896–1973)
Lipschitz modulus

\[ \text{lip}(f; \bar{x}) := \limsup_{\substack{x', x \to \bar{x}, \atop x \neq x'}} \frac{\|f(x') - f(x)\|}{\|x' - x\|}. \]

**Theorem (Hildebrand–Graves, TAMS 29: 127–153).**

Let \( X \) be a Banach space and consider a function \( f : X \to X \) and a linear bounded mapping \( A : X \to X \) which is invertible. Suppose that

\[ \text{lip}(f - A; \bar{x}) \cdot \|A^{-1}\| < 1. \]

Then \( f \) is strongly regular at \( \bar{x} \) for \( f(\bar{x}) \).

**Strong regularity:** A mapping \( F : X \rightrightarrows X \) is said to be strongly regular at \( \bar{x} \) for \( \bar{y} \) when \( (\bar{x}, \bar{y}) \in \text{gph} \ F \) and \( F^{-1} \) has a single-valued localization around \( \bar{y} \) for \( \bar{x} \) which is Lipschitz continuous.
The H-G IFT implies the classical (Dini) IFT

\[ f \text{ is strictly differentiable at } \bar{x} \iff \text{lip}(f - Df(\bar{x}); \bar{x}) = 0. \]

The classical (Dini) IFT

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be strictly differentiable at \( \bar{x} \). Then \( f \) is strongly regular at \( \bar{x} \) if and only if the derivative \( Df(\bar{x}) \) is nonsingular.
Clarke’s generalized Jacobian $\partial f(x)$

**Theorem (F. Clarke, Pac. J. Math. 64:97–102).**
Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$ which is Lipschitz continuous around $\bar{x}$ and suppose that all matrices in $\partial f(\bar{x})$ are nonsingular. Then $f$ is strongly regular at $\bar{x}$. 

Let $X$ be a Banach spaces and consider a function $f : X \to X$ which is strictly differentiable at $\bar{x}$ and any set-valued mapping $F : X \rightrightarrows X$. Let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Then $f + F$ is strongly regular at $\bar{x}$ for $\bar{y}$ if and only if the mapping

$$y \mapsto (f(\bar{x}) + Df(\bar{x})(\cdot - \bar{x}) + F(\cdot))^{-1}(y)$$

has the same property.
Theorem (A. Izmailov, MP (A) 147:581–590).

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be Lipschitz continuous around \( \bar{x} \), let \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \), and let \( \bar{y} \in f(\bar{x}) + F(\bar{x}) \). Suppose that for every \( A \in \partial f(\bar{x}) \) the mapping \( f(\bar{x}) + A(\cdot - \bar{x}) + F(\cdot) \) is strongly regular at \( \bar{x} \) for \( \bar{y} \). Then \((f + F)\) has the same property.

Lyusternik-Graves theorem (1934-1950)

**Theorem.**

Let $X$, $Y$ be Banach spaces and consider a function $f : X \to Y$ and a point $\bar{x} \in \text{int dom } f$ along with a bounded linear mapping $A : X \to Y$ which is surjective, such that

$$\text{lip}(f - A; \bar{x}) \cdot \|A^{-1}\|^- < 1,$$

where the inner “norm” of $A$ is defined as

$$\|A^{-1}\|^- := \sup_{\|y\| \leq 1} \inf_{x \in A^{-1}(y)} \|x\|.$$ 

Then $f$ is **metrically regular** at $\bar{x}$ for $f(\bar{x})$. 


Metric Regularity

A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ when $\bar{y} \in F(\bar{x})$, $\text{gph } F$ is locally closed at $(\bar{x}, \bar{y})$ and there is a constant $\tau \geq 0$ together with neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \text{for every } (x, y) \in U \times V.$$ 

The infimum of all constants $\tau \geq 0$ for which this inequality holds is the regularity modulus of $F$ at $\bar{x}$ for $\bar{y}$ denoted by $\text{reg}(F; \bar{x} \mid \bar{y})$.

Equivalent to the Aubin property of the inverse:

$$F^{-1}(x) \cap V \subset F^{-1}(x') + \tau \rho(x, x') B$$
Theorem.

Let $X$ be a complete metric space, $Y$ be a linear metric space with shift-invariant metric. Consider a mapping $F : X \rightrightarrows Y$ and a function $f : X \to Y$ such that there exist nonnegative scalars $\kappa$ and $\mu$ with

$$\kappa \mu < 1, \quad \text{reg}(F; \bar{x} | \bar{y}) \leq \kappa \quad \text{and} \quad \text{lip}(f; \bar{x}) \leq \mu.$$ 

Then $f + F$ is [strongly] metrically regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$ with

$$\text{reg}(g + F; \bar{x} | \bar{y}) \leq (\kappa^{-1} - \mu)^{-1}.$$ 

Open problem. Is there a Lyustenik-Graves theorem in nonlinear metric spaces?

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be Lipschitz continuous around \( \bar{x} \), and every \( A \in \partial f(\bar{x}) \) is surjective. Then \( f \) is metrically regular at \( \bar{x} \) for \( f(\bar{x}) \).

Extension to mapping of the form \( f + F \) acting in Banach spaces: R. Cibulka, AD and V. Veliov, (SICON 54: 3273–3296, 2016)
Bartle-Graves theorem (1952)

Let $X$ and $Y$ be Banach spaces and let $f : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ and such that the derivative $Df(\bar{x})$ is surjective. Then there is a neighborhood $V$ of $f(\bar{x})$ along with a constant $\gamma > 0$ such that $f^{-1}$ has a continuous selection $s$ on $V$ with the property

$$\|s(y) - \bar{x}\| \leq \gamma \|y - f(\bar{x})\| \quad \text{for every } y \in V.$$

Consider a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \text{gph } F$ and suppose that for some $c > 0$ the mapping $B_c(\bar{y}) \ni y \mapsto F^{-1}(y) \cap B_c(\bar{x})$ is closed-convex-valued. Consider also a function $f : X \to Y$ with $\bar{x} \in \text{int dom } f$. Let $\kappa$ and $\mu$ be nonnegative constants such that

$$\kappa \mu < 1, \quad \text{reg}(F; \bar{x} | \bar{y}) \leq \kappa \quad \text{and} \quad \text{lip}(f; \bar{x}) \leq \mu.$$ 

Then for every $\gamma > \kappa/(1 - \kappa \mu)$ the mapping $(f + F)^{-1}$ has a continuous local selection $s$ around $f(\bar{x}) + \bar{y}$ for $\bar{x}$ with the property

$$\|s(y) - \bar{x}\| \leq \gamma \|y - \bar{y}\| \quad \text{for every } y \in V.$$
A nonsmooth Bartle-Graves theorem?

**Conjecture.**

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is Lipschitz continuous around $\bar{x}$ and a convex and closed set $C \subset \mathbb{R}^n$ and suppose that for all matrices $A$ in $\partial f(\bar{x})$ the mapping

$$x \mapsto f(\bar{x}) + A(x - \bar{x}) + C$$

is metrically regular at $\bar{x}$ for $\bar{y}$. Then $(f + C)^{-1}$ has a continuous local selection around $\bar{y}$ for $\bar{x}$ which is calm at $\bar{y}$. 
Newton Method for Variational Inequalities

Variational inequality (VI): find $x \in C$ such that

$$f(x) + N_C(x) \ni 0,$$

where $N_C(x)$ the normal cone to $C$ at $x$:

$$N_C(x) = \{ w | \langle w, y - x \rangle \leq 0 \text{ for all } y \in C \}$$

Newton’s method for VI: at each step solve a linear VI:

$$f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni 0$$

Josephy (1979): If $f + N_C$ is strongly regular at $\bar{x}$ for 0 then there exists a neighborhood $O$ of $\bar{x}$ such that for every $x_0 \in O$ the method generates a unique in $O$ sequence and this sequence is superlinearly convergent to $\bar{x}$. 
Strong Regularity for Newton’s Method

Newton method for a parameterized VI

\[ x_0 = a, \quad f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p \]

Consider the mapping

\[ R^n \times R^n \ni (a, p) \mapsto \Xi(a, p) = \left\{ \{x_k\} \in l_\infty(R^n) \mid x_0 = a, \right. \]

\[ f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p, \quad k = 1, 2, \ldots \}

Theorem (with RTR (2010) and Aragon et al. (2011)).

Let \( f(\bar{x}) + N_C(\bar{x}) \ni 0 \); then \( \{\bar{x}\} \in \Xi(\bar{x}, 0) \). The mapping \( \Xi \) has a Lipschitz continuous single-valued localization around \( (\bar{x}, 0) \) for \( \{\bar{x}\} \) each value of which is a superlinearly convergent sequence to a solution \( x(p) \) of \( f(x) + N_C(x) \ni p \) if and only if \( f + N_C \) is strongly regular at \( \bar{x} \) for 0.
Open problem

Conjecture.

Let $f$ be Lipschitz continuous around $\bar{x}$ for 0 and for each $A \in \partial f(\bar{x})$ the mapping

$$x \mapsto f(\bar{x}) + A(x - \bar{x}) + N_C(x)$$

is strongly regular at $\bar{x}$ for 0. Then the mapping

$$\mathbb{R}^n \times \mathbb{R}^n \ni (a, p) \mapsto \text{the set of all sequence } \{x_k\} \in l_\infty(\mathbb{R}^n) \text{ such that } x_0 = a, \text{ and }$$

$$f(x_k) + A(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p$$

for some $A \in \partial f(x_k) \quad k = 1, 2, \ldots$, has a Lipschitz continuous single-valued localization around $(\bar{x}, 0)$ for $\{\bar{x}\}$ each value of which is a superlinearly convergent sequence to a solution $x(p)$ of $f(x) + N_C(x) \ni p$. 
Muchas Gracias!