On the Thom conjecture in $\mathbb{CP}^3$

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Some results:

- $n = 1$: [Kronheimer-Mrowka, 1994]
  the measure of complexity is the genus of the surface or equivalently b_1
- symplectic Thom conjecture: symplectic curves are genus minimizing in their homology class in symplectic 4-manifolds Morgan-Szabó-Taubes, 1996; [Ozsváth-Szabó, 2000]

Taut submanifolds

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$H_k(\mathbb{CP}^{n+1}, V_d) = 0, \quad \pi_k(\mathbb{CP}^{n+1}, V_d) = 0, \quad \text{for } k \leq n.$

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Definition

$N^{2n} \subset M^{2n+2}$ is **taut** if $\pi_k(E, \partial E) = 0$ for $k \leq n$, where E is the closed complement of a tubular neighborhood of N in M.

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- Any class $\alpha \in H_{2n}(M^{2n+2})$ has a taut representative and the homology of any such representative N is determined by $H_*(M)$ and by $b_n(N) \geq b_n(M)$. [Kato-Matsumoto, 1972]

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- If N is taut, so is $N \# \mathbb{S}^n \times \mathbb{S}^n$, where $\mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$. The signature $\sigma(N) = \sigma(\alpha)$ is determined by M and α.

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- Hence: the **complexity** $b_n(\alpha)$ of α is given by the minimal $b_n(N)$ for taut N and it satisfies $b_n(\alpha) \geq |\sigma(\alpha)|, b_n(M)$.

Thom conjecture in $\mathbb{CP}^{2m}$

Theorem (Freedman, 1977)

For any $m \geq 2$, V_d is not minimal taut in $\mathbb{CP}^{2m}$ for $d \in \mathbb{P}$, $d \neq 2, 3$ for $m = 2$, $d \neq 2$ for $m = 3$.

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He shows that V_d splits as $N \# \ell(\mathbb{S}^{2m-1} \times \mathbb{S}^{2m-1})$, where N admits a taut embedding into $\mathbb{CP}^{2m}$ homologous to V_d. Construction is via ambient surgery.

For d as in the theorem he reduces b_{2m-1} almost to the Thomas-Wood bound which comes from the G-signature theorem and realizes the bound with rationally taut submanifolds ($\pi_{2m-1}(E, \partial E)$ is d torsion) for which the same bound holds.

Thom conjecture in $\mathbb{CP}^3$

Looking for a simply connected $N_d \subset \mathbb{CP}^3$ representing $d[\mathbb{CP}^2] \in H_4(\mathbb{CP}^3)$ with minimal $b_2(N_d)$ that carries $H_2(\mathbb{CP}^3)$.

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Nonsingular V_d has:
$b_2(V_d) = d^3 - 4d^2 + 6d - 2 \sim d^3$,
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d	$b_2(V_d)$	$\sigma(V_d)$	V_d
1	1	1	$\mathbb{CP}^2$
2	2	0	$\mathbb{S}^2 \times \mathbb{S}^2$
3	7	-5	$\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$
4	22	-16	K3
5	53	-35	quintic

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Theorem (Ruberman-Slapar-S)

V_d is not minimal taut in its homology class for $d \geq 5$. There exist homologous taut submanifolds N_d with $b_2(N_d) \sim 3d^3/4$.

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The smallest $b_2(N_d)$ our method could possibly produce is $\sim d^3/2$ which yields $b_2/|\sigma| \sim 3/2$.

Sketch of proof

- Choose V_d so that a part of it carrying a large portion of H_2 can be pushed into the boundary of a 6-ball.
- Find within this part a large hyperbolic subspace of the intersection form.
- Show this subspace is supported by the sum of $\mathbb{S}^2 \times \mathbb{S}^2$'s.
- Perform ambient surgery on the spheres to reduce b_2.

Model of V_d

Start with a singular variety W_d of degree d with a single isolated singularity, e.g. $z_0 z_1^{d-1} + z_2^d = z_3^d$. Let B be a small ball about the singularity within which W_d is the cone on $W_d \cap \partial B$.

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W_d is the d-fold cyclic cover of $\mathbb{CP}^2$ branched over the singular sphere $z_0 z_1^{d-1} + z_2^d = 0$ with a unique singularity. The link of this singularity is the torus knot $T_{d-1,d}$.

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In a nearby nonsingular hypersurface V_d (given by $z_0 z_1^{d-1} + z_2^d = \varepsilon z_0^d + z_3^d$) the cone is replaced by the Milnor fibre F_d which is the d-fold cover of $\mathbb{B}^4$ branched over the Seifert surface Σ_d for the torus knot. Then $b_2(V_d) = b_2(F_d) + d$ and F_d may be pushed into ∂B.

Hyperbolic subspace in $H_2(F_d)$

The intersection form of the Milnor fibre F_d is determined by the linking form θ_d of the Seifert surface Σ_d. In fact, there is a Seifert form Θ_d for $F_d \subset \partial B = \mathbb{S}^5$ defined on $H_2(F_d) = H_1(\Sigma_d) \otimes \mathbb{Z}^{d-1}$ that satisfies $\Theta_d = \theta_d \otimes \Lambda_{d-1}$.

[Durfee-Kauffman, 1975]

Λ_k is the $k \times k$ matrix of the form $\Lambda_k = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$

Hyperbolic subspace in $H_2(F_d)$

$H_1(\Sigma_d)$ contains a subgroup G of rank $r \sim d^2/4$ such that the restriction of the Seifert form θ_d to G has the form

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & * & * & * \\ * & 0 & 0 & * & * & * \\ * & * & 0 & * & * & * \end{array}\right]$$

[Baader-Feller-Lewark-Liechti, 2015]

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Hence the restriction of the intersection form of F_d to $\widehat{G} = G \otimes \mathbb{Z}^{d-1}$ is equivalent to $\oplus_{(d-1)r/2} H$, where $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Spherical classes

The classes in $\widehat{G}$ may not be represented by spheres but by Wall's stable diffeomorphism result they are after stabilizing. For a closed 4-manifold M let $M_\ell = M\#\ell(\mathbb{S}^2 \times \mathbb{S}^2)$ be its stabilization.

Theorem (Wall, 1964)

Let M and N be simply connected closed 4-manifolds with isomorphic intersection forms. Then for some $\ell \geq 0$, M_ℓ and N_ℓ are diffeomorphic and any automorphism of the intersection form of M_ℓ is induced by a diffeomorphism.

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Choose a standard model manifold realizing the intersection form of V_d: $M_d = \frac{b_2+\sigma}{2}(\mathbb{S}^2 \times \mathbb{S}^2)\#|\sigma|\,\overline{\mathbb{CP}^2}$ for $d > 1$ odd,
$M_d = \frac{8b_2+11\sigma}{16}(\mathbb{S}^2 \times \mathbb{S}^2)\#\frac{|\sigma|}{16}K3$ for d even.
After stabilizing V_d, include the stabilizations in $\widehat{G}$ and map this into the sum of $(\mathbb{S}^2 \times \mathbb{S}^2)$'s in stabilized M_d.

Ambient surgery

Suppose $\Sigma \subset F_d \subset \partial B = \mathbb{S}^5$ is a 2-sphere with $\Sigma \cdot \Sigma = 0$. Then the normal disk bundle of Σ in $\mathbb{S}^5$ is $\Sigma \times \mathbb{B}^2 \times \mathbb{B}^1$.

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Σ bounds a properly immersed 3-disk $D \subset B$ that can be made embedded. Pairs of double points in D of opposite sign may be cancelled using the Whitney trick. The number of double points of either sign may be increased by adding kinks into Σ.

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The trivialization of the normal bundle of D may be chosen compatibly with the splitting of the normal bundle of Σ – this yields an embedded 5-dimensional 3-handle $D \times \mathbb{B}^2 \subset B$ which may be used to surger F_d along Σ. This surgery kills Σ along with its dual and preserves tautness.

Ambient surgery

To perform the above surgery procedue on two (or more) spheres Σ_1, Σ_2 their linking number in $\mathbb{S}^5$ has to be trivial – that guarantees that the corresponding disks $D_1, D_2 \subset B$ have trivial intersection number so can be made geometrically disjoint using the Whitney trick.

It follows from the structure of the Seifert form on $\widehat{G}$ that this group contains a half-dimensional subgroup with this property. For the stabilization classes this property holds by construction.