Optimal martingale transport in general dimensions

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Based on joint work with

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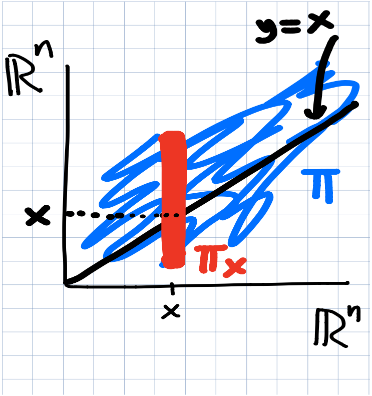
The point of this talk:

> Optimal martingale transport has rich but hidden structures, especially in multi-dimensions.

Optimal Martingale Transport Problem

- cost function $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$,
- Probability measures μ, ν on $\mathbf{R}^n$.
- $MT(\mu, \nu)$: probability measures π on $\mathbf{R}^n \times \mathbf{R}^n$ with the marginals μ, ν, and its disintegration $(\pi_x)_{x \in \mathbf{R}^n}$ has barycenter at x (martingale constraint):

$$\int y d\pi_x(y) = x.$$



Study the optimal solutions of

$$\max / \min_{\pi \in MT(\mu,\nu)} \int_{\mathbf{R}^n \times \mathbf{R}^n} c(x, y) d\pi(x, y).$$

Remark: [Strassen '65]

- $MT(\mu, \nu) \neq \emptyset$

$\Leftrightarrow$ μ and ν are in convex order;
$\mu \leq_c \nu$, i.e. $\int \xi d\mu \leq \int \xi d\nu$, $\forall$ convex $\xi : \mathbf{R}^n \to \mathbf{R}$.

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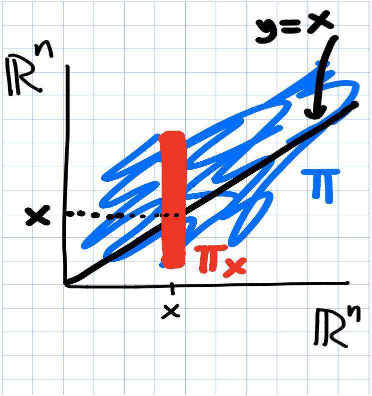

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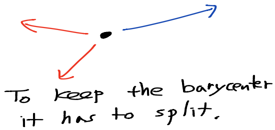
Some references:

- Discrete-time : **Beiglböck, Davis, De March, Ghoussoub, Griessler, Henry-Labordère, Hobson, Kim, Klimmek, Lim, Neuberger, Nutz, Penkner, Juillet, Schachermayer, Touzi.....**
- Continuous-time : **Beiglböck, Bayraktar, Claisse, Cox, Davis, Dolinsky, Galichon, Guo, Hu, Henry-Labordère, Hobson, Huesmann, Perkowski, Proemel, Kallblad, Klimmek, Oblój, Siorpaes, Soner, Spoida, Stebegg, Tan, Touzi, Zaev....**

Optimal Martingale Transport Problem

Existence of optimal π again follows from **weak compactness.**

[***Graphical solution (mapping solution) not available***]
π is martingale $\int y d\pi_x(y) = x$
$\Rightarrow$
for π to move a unit mass, it has to **split the mass**!



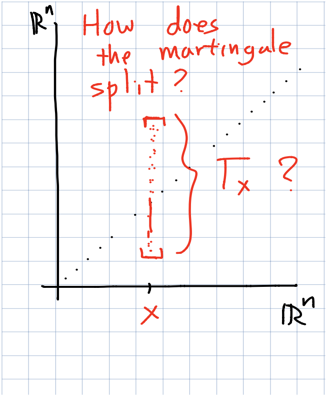
So, π **cannot** be supported on the graph $\{(x, T(x))\}$ of a map $T : \mathbf{R}^n \to \mathbf{R}^n$,
unless the trivial case $\mu = \nu$.

Optimal Martingale Transport Problem

Question: How does it split?

Let

- $\pi \in MT(\mu, \nu)$ optimal solution
- $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$: concentration set of π, i.e. $\pi(\Gamma) = 1$
- $\Gamma_x = \Gamma \cap (\{x\} \times \mathbf{R}^n)$ the vertical slice at x (the **"Splitting set"**)



Question :

- What is the structure of π, or the set Γ, especially Γ_x?
- When is π unique?

From now on, we will focus on the case:

- $\mu \ll$ *Lebesgue*.
-

$$c(x, y) = |x - y|$$

1-dimensional results

Theorem (Hobson-Neuberger '13, Beiglböck-Juillet '13)

Suppose $n = 1$ *and*

- $c(x, y) = |x - y|$
- $\mu \leq_C \nu$ *on* $\mathbf{R}$ *and* $\mu << \mathcal{L}^1$.
- $\pi \in MT(\mu, \nu)$ *optimal solution (for max / min).*
- *Assume* $\mu \wedge \nu = 0$ *for the minimization problem.*

Then

- *There exists* $\Gamma \subset \mathbf{R} \times \mathbf{R}$*: concentration set of* π*, i.e.* $\pi(\Gamma) = 1$*, such that for a.e.* $x \in \mathbf{R}$*,*

$$\#(\Gamma_x) \leq 2, \quad \text{for } \Gamma_x = \Gamma \cap (\{x\} \times \mathbf{R}),$$

that is, the disintegration (conditional probability) π_x *is concentrated on at most two points.*

- *In particular, the optimal solution* π *is unique.*

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Higher dimensions?

Theorem (Dimension reduction. Ghoussoub, K. & Lim)

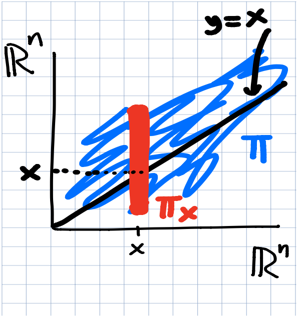
Assume:

- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^n$
- $\pi \in MT(\mu, \nu)$ *be optimal.*

Then the following holds:

- *There is concentration set of* π, $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$ *such that*

$$\dim(\Gamma_x) \leq n - 1 \quad \text{for } \mu\text{-almost every } x,$$



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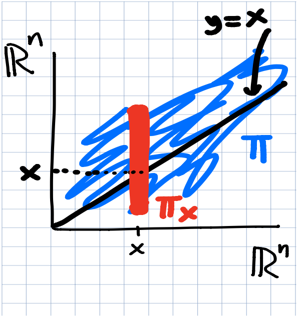


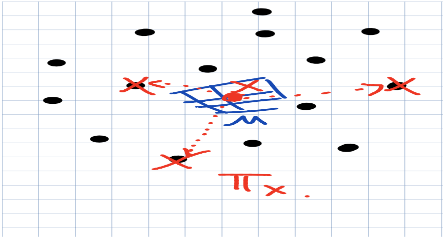
Theorem (Discrete target. Ghousshoub, K. & Lim)

If *furthermore,* ν *is discrete* $\nu = \sum_{k=1}^{\infty} q_i \delta_{y_i}$,

then *for* μ *a.e.* x*, under the optimal martingale transport,*

$$x \mapsto \; n+1 \text{ vertices of a } n\text{-dimensional simplex in } \mathbf{R}^n.$$

Moreover. the optimal solution is unique.



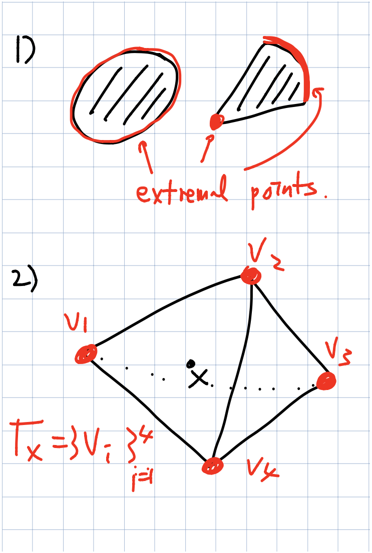
Conjectures in higher dimensions. [Ghousshoub, K. & Lim]

Assume:

- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^n$
- $\pi \in MT(\mu, \nu)$ be optimal.

Conjecture: Then, $\exists$ concentration set Γ, such that for μ almost every x,

$$\overline{\Gamma}_x = \mathrm{Ext}\left(\mathrm{conv}(\overline{\Gamma}_x)\right).$$



Progress towards the conjecture

Assume:

- $c(x, y) = |x - y|$
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- $\pi \in MT(\mu, \nu)$ be optimal.

Conjecture: Then, $\exists$ concentration set Γ, such that for μ almost every x,

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Theorem (Ghoussuob, K. & Lim)

Conjecture 1 holds in the following cases:

- *$n = 2$, or*
- *ν is obtained from μ by diffusion with respect to a time-dependent elliptic operator. More generally, if there is a stopping time $T > 0$ of a Brownian motion with $B_0 \sim \mu$ and $B_T \sim \nu$.*

Key principle

- Duality

Duality

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  - $$\inf_{\pi \in MT(\mu,\nu)} \int c(x,y) d\pi(x,y)$$
    $$= \sup\{\int \beta(y) d\nu(x) - \int \alpha(x) d\mu(x) \ :$$
    $$\beta(y) \leq c(x,y) + \alpha(x) + \gamma(x) \cdot (y-x), \ \ \forall x, y\}$$
  - $$\sup_{\pi \in MT(\mu,\nu)} \int c(x,y) d\pi(x,y)$$
    $$= \inf\{\int \beta(y) d\nu(x) - \int \alpha(x) d\mu(x) \ :$$
    $$\beta(y) \geq c(x,y) + \alpha(x) + \gamma(x) \cdot (y-x), \ \ \forall x, y\}$$
- If the maximizer/minimizer (α, β, γ) exists, then the set,

  saturation set: $\Gamma = \{(x,y) \mid \beta(y) = c(x,y) + \alpha(x) + \gamma(x) \cdot (y-x)\}$

  gives a concentration set of an optimal π.
  In this case, we say "π admits a dual".

Question Can one always have a dual (α, β, γ) for an optimal π?

Answer

No! [Beiglböck-Juillet '13]

Counterexample: For the maximization problem,

$\mu = \nu$ cannot attain dual (Exercise: Otherwise, γ must be $\pm\infty$ on $[0, 1]$.)

(The term $\gamma(x) \cdot (y - x)$ is the trouble maker.)

We do not know for the minimization problem in general.

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We do not know for the minimization problem in general.

There are cases where dual functions exist

Theorem (Ghoussoub, K., & Lim)

The dual functions (locally) exist for an optimal $\pi \in MT(\mu, \nu)$ if

- $\mu << Leb$*, compactly supported*
- ν *is obtained from* μ *by diffusion with respect to a time-dependent elliptic operator. More generally, if there is a stopping time* $T > 0$ *of a Brownian motion with* $B_0 \sim \mu$ *and* $B_T \sim \nu$.

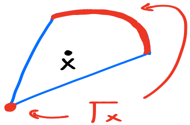
It is good to have dual functions.

If dual functions are attained.

Lemma (Ghoussoub, K. & Lim '15)

Let $c = |x - y|$. Suppose a dual (α, β, γ) is attained and Γ its saturation set.
Then *for a.e. x*

$$\overline{\Gamma}_x = \mathrm{Ext}\left(\mathrm{conv}(\overline{\Gamma}_x)\right).$$



Proof.
" Differentiate the duality relation to get information!"

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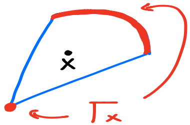

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Proof continued.

duality relation (for the minimization problem)

$$\beta(y) \le c(x,y) + \alpha(x) + \gamma(x)\cdot(y-x)\ \forall x \in X_\Gamma, y \in Y_\Gamma,$$
$$\beta(y) = c(x,y) + \alpha(x) + \gamma(x)\cdot(y-x)\ \forall (x,y) \in \Gamma.$$

If $(x,y) \in \Gamma$,

$$|x-y| + \gamma(x)\cdot(y-x) + \alpha(x) \le |x'-y| + \gamma(x')\cdot(y-x') + \alpha(x') \quad \forall x'$$
$$\Rightarrow \nabla_x(|x-y| + \gamma(x)\cdot(y-x) + \alpha(x))$$
$$= \frac{x-y}{|x-y|} + \nabla\gamma(x)\cdot(y-x) - \gamma(x) + \nabla\alpha(x) = 0.$$

Now suppose that we can find $\{y, y_0, ..., y_s\} \subset \overline{\Gamma}_x$ with $y = \Sigma_{i=0}^s p_i y_i$, $\Sigma_{i=0}^s p_i = 1$, $p_i > 0$. Then we get

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But this can hold only if all y_i lie on the same ray emanated from x. Hence... □

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Summary: the conclusion under dual attainment

Theorem (Ghousshoub, K. & Lim '15)

Let

- $c(x, y) = |x - y|$,
- $\mu << \mathcal{L}^n$,
- $\pi \in MT(\mu, \nu)$: *optimal solution for martingale transport problem.*

Suppose that π *admits a dual* (α, β, γ). *Let*

$$\Gamma = \{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d \,|\, \beta(y) = c(x, y) + \alpha(x) + \gamma(x) \cdot (y - x)\}.$$

Then Γ *is a concentration set of* π, *(i.e.* $\pi(\Gamma) = 1$*), and for* μ *a.e.* x,

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For general cases where we do not have dual functions:

Partition:
Make partition into duality attainable components!

For general cases where we do not have dual functions:

Theorem (Beiglböck-Juillet '13)

Suppose

- $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ *continuous.*
- $\pi \in MT(\mu, \nu)$*: an optimal solution for martingale transport problem.*

Then *there exists a concentration set* $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$*, (i.e.* $\pi(\Gamma) = 1$*)*
such that Γ *is* **monotone***,*
that is, any **finite subset** $H \subset \Gamma$ *admits a dual.*

Partition into dual attainable components.

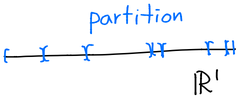
Theorem (Beiglböck-Juillet '13 for 1dim, Ghousshoub, K. & Lim '15 for general dim)

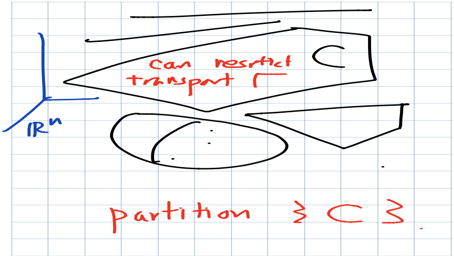
Suppose

- $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ *continuous.*
- $\pi \in MT(\mu, \nu)$*: an optimal solution for martingale transport problem.*

Then *there exists a concentration set* $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$, *(i.e.* $\pi(\Gamma) = 1$*):*

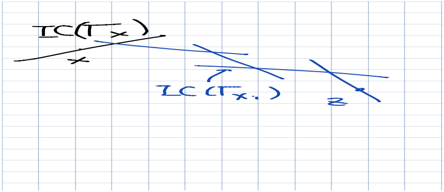
- *One can define mutually disjoint convex sets* $\{C\}$
- *such that "transport"* Γ *is partitioned on* C*'s,*
- *and on each such component* C*, the set* Γ *attains a dual.*





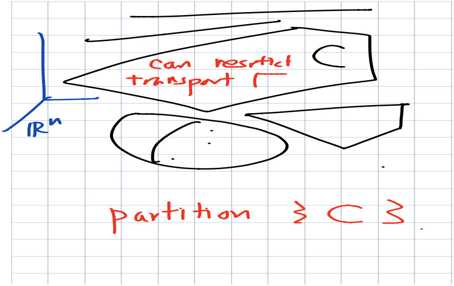
Convex Partition in n-dimensions

- $x \sim_1 z$ equivalence relation if there is a chain of $IC(\Gamma_x) := int(conv\Gamma_{x_i})$'s



- Get partition for $\sim_1$.
  Rmk: In 1-dim, we can stop here.
- Take convex hull for each component of $\sim_1$.
- Define equivalence relation $\sim_2$ using chains of those convex hulls
- Iterate this procedure on and on,
- to get equivalence relation $\sim$ and corresponding "convex" partition $\{C\}$ generated by Γ.
- It can be shown (highly nontrivial) that each such component C attains dual!

Now, for each such component, dual is attained.



The method of [Ghousoub, K. & Lim '15]:

Disintegrate μ and ν into partition $\{C\}$, each of which attains dual.

If the disintegration of μ on each C is **absolutely continuous**,

to use the dual functions and their a.e. differentiability to get the structural result for μ-a.e. x.

Partition can be useful **only if** we know good disintegration of μ along it.

But unfortunately, getting such a **good disintegration** is NOT clear in general.

Nikodym set [Ambrosio, Kirchheim, and Pratelli '04]
There is a Nikodym set in $\mathbf{R}^3$,

- having full measure in the unit cube,
- intersecting each element of a family of pairwise disjoint open lines
- only at one point.

This means, the point where we have differentiability of dual may not, in general, belong to the set we want.

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Still can handle dimension question even without good disintegration:

Corollary (Ghoussoub, K. & Lim '15)

Suppose

- $c(x, y) = |x - y|$
- $\pi \in MT(\mu, \nu)$ *optimal*
- $\mu << \mathcal{L}^n$.

Then, *there is a concentration set* Γ *of* π*, such that for* μ*-almost every* x*,*

$$\dim \Gamma_x \leq n - 1.$$

Proof.

- If $\dim C = n$, then C is open, thus, μ can be restricted on C, so absolutely continuous on C! Apply previous results.
- For other components with $\dim C \leq n - 1$, but, in this case already the dimension is $\leq n - 1$.

□

A case with good disintegration: discrete target, thus countable partition components

Theorem (Discrete target. Ghousshoub, K. & Lim '15)

If *furthermore,* ν *is discrete* $\nu = \sum_{k=1}^{\infty} q_i \delta_{y_i}$,

then *for* μ *a.e. x, under the optimal martingale transport,*

$$x \mapsto \; n+1 \text{ vertices of a } n\text{-dimensional simplex in } \mathbf{R}^n.$$

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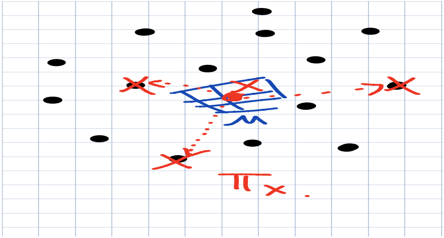

A case with good disintegration: two dimensions

Theorem (Ghoussoub, K. & Lim '15 $n = 2$)

Suppose

- $c(x, y) = |x - y|$,
- $\pi \in MT(\mu, \nu)$ *optimal,*
- $\mu << \mathcal{L}^n$,
- $n = 2$,

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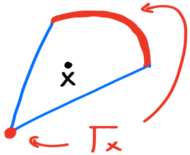
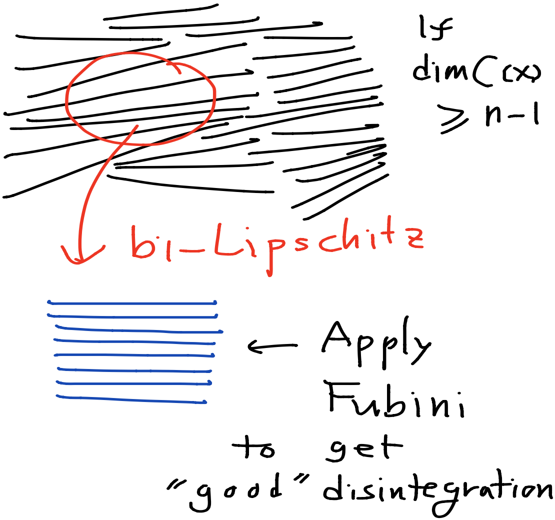
$$\overline{\Gamma}_x = \mathrm{Ext}(\mathrm{conv}(\overline{\Gamma}_x)).$$



Codimension ≤ 1 case. **Idea:** Flattening!



Summary:

To study the structure of optimal martingale transport in $MT(\mu, \nu)$ with $\mu << \mathcal{L}^n$ in **general dimensions** n:

- Find **optimal** martingale plan $\pi \in MT(\mu, \nu)$ using compactness.
- Get a suitable **monotone set** Γ.
- Apply the **partition** of Γ into **duality** attainable components C.
- Get **dual functions** α, β, γ for Γ in C.
- **Almost everywhere differentiability** of α, γ on C.

If μ disintegrates into an **absolutely continuous measure** μ_C on each component C,

- Get the **structure** of Γ (of Γ_x for μ_C a.e. x) in each C from **almost everywhere differentiability**,
- thus finally, get the **structure** of Γ (of Γ_x for μ-a.e. x)!

Summary:

To study the structure of optimal martingale transport in $MT(\mu,\nu)$ with $\mu << \mathcal{L}^n$ in **general dimensions** n:

- Find **optimal** martingale plan $\pi \in MT(\mu,\nu)$ using compactness.
- Get a suitable **monotone set** Γ.
- Apply the **partition** of Γ into **duality** attainable components C.
- Get **dual functions** α, β, γ for Γ in C.
- **Almost everywhere differentiability** of α, γ on C.

If μ disintegrates into an **absolutely continuous measure** μ_C on each component C,

- Get the **structure** of Γ (of Γ_x for μ_C a.e. x) in each C from **almost everywhere differentiability**,
- thus finally, get the **structure** of Γ (of Γ_x for μ-a.e. x)!

Some related work:

- **[Beiglböck, Nutz, & Touzi '15]** : quasi-sure duality.
- **[De March& Touzi '17] [Oblój & Siorpaes '17]**: canonical partition for martingale transport.

Thank You Very Much!