Volatility derivatives in (rough) forward variance models

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Reminders on forward variances

- Forward variance V_t^T are fair strikes of variance swaps :

$$\text{payoff of Var swap over } [t, T] = \frac{1}{T-t} \sum_{t_i \in [t,T]}^{N} \big(\log(S_{t_{i+1}}) - \log(S_{t_i})\big)^2 - V_t^T$$

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- Take $T_2 > T_1$. By combining positions in var swaps over $[t, T_2]$ and $[t, T_1]$, we construct the payoff

$$\frac{1}{T_2 - T_1} \sum_{t_i \in [T_1, T_2]} \left(\log(S_{t_{i+1}}) - \log(S_{t_i}) \right)^2 - V_t^{T_1, T_2}$$

  where

$$V_t^{T_1,T_2} = \frac{(T_2 - t) V_t^{T_2} - (T_1 - t) V_t^{T_1}}{T_2 - T_1}$$

  is the forward variance over $[T_1, T_2]$.

Forward variances can be traded

- By entering in the opposite positions in variance swaps at a date $t' \geq t$, we remove the realized variance part.
- We materialize a position depending only on forward variances :

$$\text{portfolio value at } T_2 = V_{t'}^{T_1,T_2} - V_t^{T_1,T_2}$$

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  Forward variances can be traded at zero cost.

- ▶ Under a pricing measure, the $(V_t^{T_1,T_2})_{0 \leq t \leq T_1}$ have to be martingales

Instantaneous forward variance ξ_t^T

- Define **instantaneous forward variance** by

$$\xi_t^T = \frac{d}{dT}\left((T-t)V_t^T\right), \qquad t < T$$

so that

$$V_t^T = \frac{1}{T-t}\int_t^T \xi_t^u \, du, \qquad t < T$$

and

$$V_t^{T_1,T_2} = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \xi_t^u \, du, \qquad t < T_1 < T_2.$$

- Note that if Δ is small, then

$$V_t^{T,T+\Delta} \approx \xi_t^T$$

A class of models based on Gaussian processes

$$\xi_t^T = \xi_0^T \exp\left(\int_0^{t} K(T-s) \cdot dW_s - \frac{1}{2} \int_0^{t} K(T-s) \cdot \rho K(T-s) ds \right) \qquad t \leq T$$

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- $(\xi_0^T)_{T \ge 0}$ is the initial forward variance curve – a market parameter.
- W is a Brownian motion in $\mathbb{R}^n$ with correlation matrix ρ, and

$$\int_0^t K(T-s) \cdot dW_s = \sum_{i=1}^n \int_0^t K_i(T-s) dW_s^i$$

$$\int_0^t K(T-s) \cdot \rho K(T-s) ds = \sum_{i,j=1}^n \int_0^t K_i(T-s) \rho_{i,j} K_j(T-s) ds$$

- Deterministic kernels $K_i \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+^*)$.

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- For every T, $(\xi_t^T)_{t\le T}$ is the solution of the SDE

$$\xi_t^T = \xi_t^T K(T-t)\cdot dW_t, \qquad t \le T$$

- Does not belong to the affine family.
- Interest for simulation/calibration : only Gaussian r.v. are involved.

Choice of kernels in practice : $\tau \mapsto K(\tau)$ decreasing.

Parametric examples (I)

- Bergomi's model [Bergomi 05], [Dupire 93] with $n = 1$ factor
$$K(\tau) = \omega\, e^{-k\tau}$$
with $\omega, k > 0$.
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$$\begin{aligned}
\xi_t^T &= \xi_0^T\, \mathcal{E}\left(\omega \int_0^t e^{-k(T-s)} dW_s\right) \\
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&= \xi_0^T\, \exp\left(K(T-t) X_t - \frac{1}{2}\int_0^t K(T-s)^2 ds\right)
\end{aligned}$$
where X is the OU process $dX_t = -k\, X_t + dW_t$.

- For every t, $\xi_t^T = \Phi(T-t, X_t)$: the forward variance curve $\xi_\cdot^T$ is a function of one single Markov factor X.

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- Bergomi's n-factor model [Bergomi 05] is the n-dim extension :

$$K_i(\tau) = \omega_i\, e^{-k_i\tau}$$

Parametric examples (II)

- The *rough* Bergomi model of [Bayer, Friz, Gatheral 2016] :

$$K(\tau) = \frac{\omega}{\tau^{\frac{1}{2}-H}}, \qquad H \in (0, 1/2)$$

so that

$$\xi_t^T = \xi_0^T \exp\left(\omega \int_0^t \frac{1}{(T-s)^{\frac{1}{2}-H}} dW_s - \frac{1}{2}\omega^2 \int_0^t \frac{1}{(T-s)^{1-2H}} ds\right)$$

▶ Do not have a low-dimensional Markovian representation of the curve

$$T \mapsto (\xi_t^T)_{T \geq t}$$

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▶ Do not have a low-dimensional Markovian representation of the curve

$$T \mapsto (\xi_t^T)_{T \geq t}$$

- For the moment (in this presentation), nothing in this model is rough.

For every T, the processes

$$(\xi_t^T)_{t \leq T} \quad \text{are martingales}$$

Constructing a consistent model for S_t

Reminders : in a general stochastic volatility model

$$dS_t = \mu \, S_t dt + \sigma_t \, S_t \, dW_t^{\mathrm{hist}}$$

- Realized variance can be replicated with the underlying + a log-contract

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- Realized variance can be replicated with the underlying + a log-contract
  - Indeed, by Itô's formula applied to $\log(S)$

$$\frac{1}{T-t} \langle \log S \rangle_{[t,T]} = \frac{1}{T-t} \int_t^T \sigma_u^2 du = \frac{2}{T-t} \left(-\log \frac{S_T}{S_t} + \int_t^T \frac{1}{S_u} dS_u \right)$$

    Almost sure replication of $\langle \log S \rangle_{[t,T]}$

  - This yields (taking interest rates to be zero)

$$V_t^T = \mathrm{price}_t \left(\frac{1}{T-t} \int_t^T \sigma_u^2 du \right) = \mathrm{price}_t \left(-\frac{2}{T-t} \log \frac{S_T}{S_t} \right)$$

A consistent model for S_t

Given instantaneous forward variances ξ_t^T

- The model

$$dS_t = S_t \sqrt{\xi_t^t}\, dZ_t$$

  where Z is a Brownian motion, is consistent with the given ξ_t^T

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In the sense : the price of the log-contract in this model is

$$\begin{aligned}\mathrm{price}_t\left(\frac{-2}{T-t}\log\frac{S_T}{S_t}\right) &= \mathbb{E}\left[\frac{1}{T-t}\int_t^T \xi_u^u\, du \Big| \mathcal{F}_t\right] \\ &= \frac{1}{T-t}\int_t^T \mathbb{E}\left[\xi_u^u|\mathcal{F}_t\right] du = \frac{1}{T-t}\int_t^T \xi_t^u du\end{aligned}$$

where $\mathcal{F}_t = \mathcal{F}_t^{W,Z}$

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where $\mathcal{F}_t = \mathcal{F}_t^{W,Z}$

▶ Hedging of European options on S with underlying + forward variances

Rough Bergomi model, again

- To see what is rough in rough Bergomi

  we have to look at the consistent model for S :

$$dS_t = S_t \sqrt{\xi_t^t}\, dZ_t$$

The instantaneous volatility ξ_t^t of S is rough because

$$\xi_t^t = \exp\left(\omega\, x_t^t - \frac{1}{2}\omega^2 \int_0^t \frac{1}{(t-s)^{1-2H}} ds\right)$$

and

$$x_t^t = \int_0^t \frac{1}{(t-s)^{\frac{1}{2}-H}}\, dW_s$$

is a Volterra process which admits a β-Hölder modification for $\beta < H$

The VIX index

- The VIX is the price of the log-contract with 30 days maturity written on the SP500 :

$$\mathrm{VIX}_t := \sqrt{\mathrm{mkt\ price}_t\left(-\frac{2}{\Delta}\log\frac{S_{t+\Delta}}{S_t}\right)} \qquad \text{where } \Delta = 30\text{days}$$

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- The value of VIX is quoted by the Chicago Option Exchange, by static replication of the payoff $\log(S)$:

$$\mathrm{VIX}_t = \sqrt{\frac{2}{\Delta}\left(\int_0^{S_t}\frac{1}{K^2}\mathrm{P}_t(t+\Delta,K)dK + \int_{S_t}^{\infty}\frac{1}{K^2}\mathrm{C}_t(t+\Delta,K)dK\right)}$$

where $\mathrm{P}_t(T,K)$ and $\mathrm{C}_t(T,K)$ are market prices of put and call options on S, observed at t.

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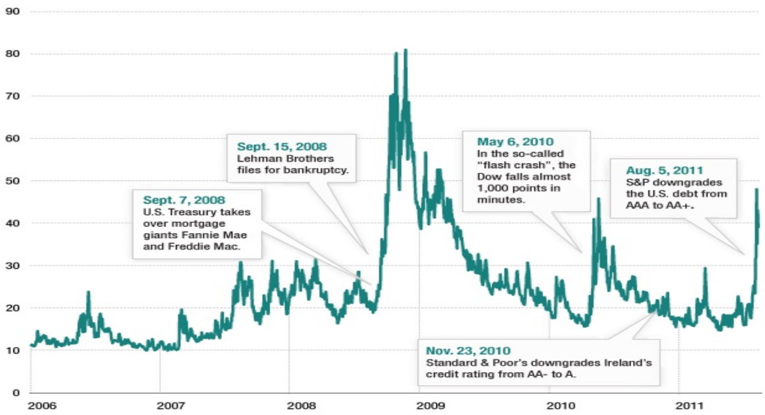
- Is VIX an implied volatility ? Yes, it is precisely the implied volatility of the log-contract.

History of VIX (2006-2011)



VIX in a stochastic volatility model

- In general, VIX and forward variances of variance swaps *do not* coincide

$$\mathrm{VIX}_t^2 \neq V_t^{t+\Delta} = \frac{1}{\Delta} \int_t^{t+\Delta} \xi_t^u \, du$$

because the replication of variance swaps with log-contracts is only approximate in practice.

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- Within a stochastic volatility model, on the contrary

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- Consequence : in general, we will be able to calibrate a forward variance model $(S_t, \xi_t^{\cdot})$ to at most 2 of the 3 different markets :
    - VIX market
    - SP500 options market
    - Variance swap market on SP500

Pricing of VIX derivatives at $t = 0$

The price at $t = 0$ of a VIX option with payoff φ is

$$\mathbb{E}\left[\varphi(\mathrm{VIX_T})\right] = \mathbb{E}\left[\varphi\left(\sqrt{V_T^{T+\Delta}}\right)\right] = \Psi(0, \xi_0^{\cdot})$$

where

$$\Psi(0, x^{\cdot}) = \mathbb{E}\left[\varphi\left(\left(\frac{1}{\Delta}\int_T^{T+\Delta} x^u \, e^{\int_0^T K(u-s)\cdot dW_s - \frac{1}{2}h(0,T,u)} du\right)^{1/2}\right)\right]$$

and $h(t, T, u) = \int_t^T K(u-s) \cdot \rho K(u-s) ds$.

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- If Markov repr (e.g. classical Bergomi), $\int_0^T K(u-s) \cdot dW_s = K(u-T)\, X_T$
- Otherwise : finite point $(u_i)_{i=1,\dots,N}$ quadrature formula + simulation of the correlated Gaussian vector

$$\left(\int_0^T K(u_1, s) \cdot dW_s, \dots, \int_0^T K(u_N, s) \cdot dW_s\right)$$

$\rightsquigarrow$ see A. Jacquier's talk for rates of convergence.

Term structure of volatility of volatility

- Denote

$$\hat{\sigma}(t, T)$$

the at-the-money implied volatility of an option on the forward volatility $\sqrt{V_t^T}$.

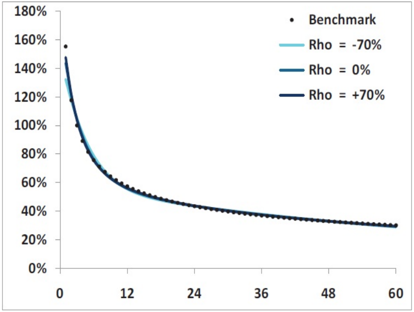
Proposition (ATM implied volatility of forward volatility)

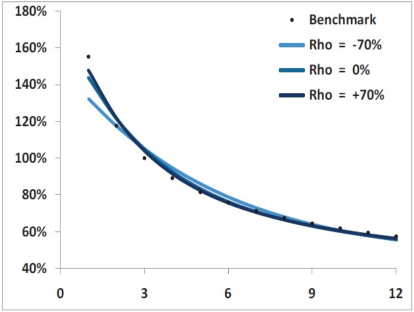
The following asymptotics hold : for every T

$$\hat{\sigma}(t, T) \underset{t \to 0}{\longrightarrow} \hat{\sigma}(0, T) := \frac{1}{2 \int_0^T \xi_0^u du} \sqrt{\int_0^T \xi_0^u \, K(u) \cdot \rho \int_0^T \xi_0^{u'} \, K(u') du'}$$

- By choosing the kernels K, we can reach a prescribed target behavior of $\hat{\sigma}(0, T)$

Term structure of volatility of volatility





- Black dots : target behavior for $\hat{\sigma}(0, T)$, as a function of T (months).
- Very well described by a power law $\frac{1}{T^\alpha}$, $\alpha \approx 0.4 - 0.5$

Term structure of volatility of volatility

- Choice 1 : $n = 1$ power kernel $K(u) = \frac{\omega}{u^{\frac{1}{2}-H}}$

  Then, if $u \mapsto \xi_0^u$ is constant,

$$\hat{\sigma}(0, T) = \frac{\text{const.}}{T^{\frac{1}{2}-H}}$$

  which is exactly our target term-structure, when $H \approx 0.1$.

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- Choice 2 : $n = 2$ exponential kernels

$$K_i(u) = \omega\, e^{-k_i u} \qquad \text{and } d\langle W^1, W^2\rangle_t = \rho\, dt$$

  with $k_1 \ll 1$, $k_2 \gg 1$.

  The resulting behavior of $\hat{\sigma}(0, T)$ is shown by the blue curves

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- A model with fractional kernel reaches the target behavior with $n = 1$ factor and two parameters ω, H.
- A classical Bergomi model does this with $n = 2$ factors and four parameters k_1, k_2, ρ, ω.

An extended class of forward variance models

As mentioned by Antoine, in the class of models above

- The ξ_t^T are log-normal. Forward variances $\frac{1}{\Delta}\int_T^{T+\Delta} \xi_T^u du$ are close to log-normal.
- Incapability of generate a reasonable smile for VIX options.

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▶ Inspired by [Bergomi 2008], we set

$$\xi_t^T = \xi_0^T\, f^T(t, x_t^T)$$

where x_t^T denotes our Gaussian factor

$$x_t^T = \int_0^t K(T-s) \cdot dW_s$$

and the $f^T(\cdot,\cdot)$ are smooth functions to be determined.

An extended class of forward variance models

- We need to impose some conditions on f^T :
  - $f^T(t,x) \geq 0$
  - Initial condition $\xi_0^T \;\Rightarrow\; = f^T(0,0) = 1, \, \forall T$
  - $(\xi_t^T)_{0 \leq t \leq T}$ needs to be martingale :

$$d\xi_t^T = \Big(\partial_t f^T(t, x_t^T) + \frac{1}{2} K \cdot \rho K \; \partial_{xx} f^T(t, x_t^T)\Big) dt + \partial_x f^T(t, x_t^T) dx_t^T$$

Therefore, we require that the $f^T(\cdot)$ solve the family of PDE

$$\partial_t f^T(t,x) + \frac{1}{2} K(T-t) \cdot \rho K(T-t) \partial_{xx} f(t,x) = 0, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

An extended class of forward variance models

▶ A simple representation : any $C^{1,2}([0,T) \times \mathbb{R})$ function f^T with exponential growth satisfying the PDE above can be written in terms of its terminal condition

$$f^T(t,x) = \mathbb{E}\left[f^T(T, x + \sqrt{h(t,T,T)}\, G)\right]$$

where G is a standard Gaussian random variable (and recall that $h(t,T,T) = \int_t^T K(T-s) \cdot \rho K(T-s) ds$).

- Positive solutions $f^T(\cdot,\cdot)$ are parametrized by positive final conditions $f^T(T,\cdot)$
- We can generate several parametric families of solutions.

Parametric choice 1 : polynomials

- The terminal condition :

$$f^T(T, y) = a(T)y^2 + b(T)y + c(T)$$

leads to a quadratic Gaussian model

$$\xi_t^T = \xi_0^T f^T(t, x_t^T) = \xi_0^T \Big(a(T) \left[(x_t^T)^2 - h(t, T) \right] + b(T) x_t^T + 1 \Big)$$

where $h(t, T) = \int_0^t K(T - s) \cdot \rho K(T - s) ds$

▶ We are free to choose $a(T)$, $b(T)$ s.t. $1 - a(T)h(T, T) - \frac{b(T)^2}{4a(T)} \geq 0$ (positivity condition).

- Example : if $b(T) = 0$, ξ_t^T has a χ^2 distribution.

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- The more general terminal condition :

$$f^T(T,y) = \sum_{k=0}^{n} a_k^T y^{2k}$$

leads to polynomial functions $x \mapsto f^T(t,x)$.

Parametric choice 2 : exponentials

- The terminal condition :

$$f^T(T,y) = \sum_{k=1}^{m} \gamma_k \; e^{\omega_k \, y} \qquad \text{where} \qquad \sum_{k=1}^{m} \gamma_k = 1, \qquad m \in \mathbb{N}$$

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- With this choice, forward variances

$$\xi_t^T = f^T(t, x_t^T)$$

are sums of log-normals which can be made very different from a single log-normal

- We have expressions and numerical methods for VIX derivatives similar to the previous case (where $m = 1$).

A simple version of the rough model where forward variances are not log-normal

Rough_model ($n = 1, m = 2$) : $n = 1$ gaussian factor, and $m = 2$ basis functions

$$f^T(t,x) = (1-\gamma^T)\exp\left(\omega_1^T x - \frac{1}{2}(\omega_1^T)^2 h(t,T)\right) + \gamma^T \exp\left(\omega_2^T x - \frac{1}{2}(\omega_2^T)^2 h(t,T)\right)$$

$$\xi_t^T = \xi_0^T f^T(t, x_t^T)$$

$$x_t^T = \int_0^t K(T-s)dW_s, \qquad K(T-s) = \frac{1}{(T-s)^{\frac{1}{2}-H}}$$

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- This model depends on the global parameter

$$H$$

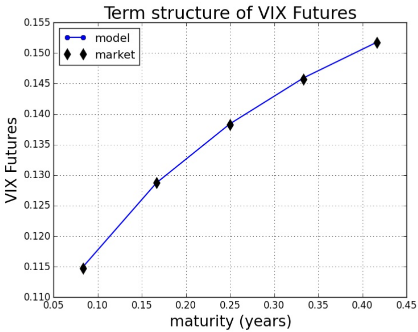
and on the four term-structure parameters

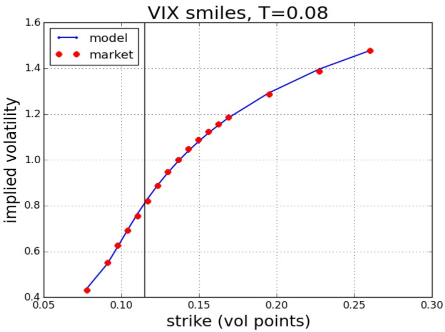
$$\xi_0^T,\ \gamma^T,\ \omega_1^T,\ \omega_2^T$$

which we can use to fit an initial term-structure of VIX Futures and the smiles of VIX options.

Calibration to VIX market ($m = 2$ exponential fcts)

VIX Futures (left) and VIX implied volatilities (right) on 22 Nov 2017, T = 20 Dec





$$H = 0.1 \text{ (fixed)} \quad \xi_0^u|_{T \le u \le T+\Delta} = 0.0145 \quad \gamma = 0.689 \quad \omega_1 = 2.074 \quad \omega_2 = 0.215$$

Non-parametric choices of f leading to exact calibration

are possible

Conclusion & further directions

- The consistent model for the SP500 :

$$dS_t = S_t \sqrt{\xi_t^t}\, dZ_t$$

might be a good candidate for a joint calibration of VIX and SP500 options

- See the talk of J. Guyon at QuantMinds conference 2018 (former Global Derivatives), taking place this week, for some considerations about the feasibility of this joint calibration.

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In summary :

- Volterra Gaussian processes offer a considerable flexibility in the modeling of forward variances.
- Using more general functions than single exponentials allows to accomodate smiles of options on VIX, while keeping the Gaussian framework.
- "Rough" power kernels inevitably make the pricing of VIX Futures & options less tractable.
- Still accessible via Monte-Carlo + variance reduction.

Thank you for your attention

Direct modeling of VIX Futures

Instead of instantaneous forward variances ξ_t^T, we can apply the framework above to model VIX Futures $(\mathrm{FVIX}_t^i)_{t \le T_i}$ directly :

$$\mathrm{FVIX}_t^i = \mathrm{FVIX}_0^i \, f^i\big(t, x_t^{T_i}\big) \qquad\qquad T_i = \text{VIX maturities}$$

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- This opens the way to **non-parametric choices** of f^i :
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  - Which we can exactly fit with the distribution of

    $$\mathrm{FVIX}_{T_i}^i = \mathrm{FVIX}_0^i \, f^i\big(T_i, x_{T_i}^{T_i}\big)$$

    by choosing a monotone terminal function $f^i(T_i, \cdot)$ (and using the fact that $x_{T_i}^{T_i}$ is Gaussian).