Modified trace and logarithmic invariants

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Motivation and context

- Witten-Reshetikhin-Turaev TFT uses partially representations of the quantum group.
  Only simple modules with non vanishing quantum dimension are used.
- Geer-Patureau-Virelizier, Geer-Patureau-Kujawa:
  - Trace on ideal in pivotal category, called *modified* trace.
  - Invariants for links colored with projective objects.
- Costantino-Geer-Patureau, B-Costantino-Geer-Patureau:
  - Non semi-simple invariants and TFT from unrolled quantum $sl(2)$.
  - The category is graded, finite in each degree up to action of invertible objects, generically semisimple.
  - De Renzi: generalisation and 2-functor extension.
- What about non semisimple finite case e.g. restricted quantum $sl(2)$?

Papers

- Recent contribution with coauthors

  BBGa B-Beliakova-Gainutdinov, *Modified trace is a symmetrised integral* , arXiv:1801.00321.
  Structural results in finite dimensional Hopf algebra H which determine modified trace on H-pmod.

  BBGe B-Beliakova-Geer, *Logarithmic Hennings invariants for restricted quantum sl(2)* , arXiv:1705.03083.

- Further contribution from our friends

  GPDR Geer-Patureau-De Renzi, *Renormalized Hennings Invariants and 2+1-TQFTs*, arXiv:1707.08044.

Notation

- $H = (H, \Delta, \epsilon, S)$ is a finite dimensional Hopf algebra over a field $\Bbbk$.
- For $V \in H$-mod, $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ with $(hf)(x) = f(S(h)x)$.
  We have the standard left duality morphisms

$$\mathrm{ev}_V : \ V^* \otimes V \to \Bbbk, \qquad \text{given by} \quad f \otimes v \mapsto f(v),$$
$$\mathrm{coev}_V : \ \Bbbk \to V \otimes V^*, \qquad \text{given by} \quad 1 \mapsto \sum_{j \in J} v_j \otimes v_j^*.$$

  Here $\{v_j \,|\, j \in J\}$ is a basis of V and $\{v_j^* \,|\, j \in J\}$ is the dual basis of V^*.

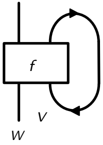
Pivotal Hopf algebra

- A pivot is a group-like $\boldsymbol{g} \in H$ such that $\forall x \ S^2(x) = \boldsymbol{g} x \boldsymbol{g}^{-1}$.
  $(H, \boldsymbol{g})$ is a pivotal Hopf algebra.
- If $(H, \boldsymbol{g})$ is pivotal, then H-mod is a pivotal category.
  Right duality morphisms are defined as follows

$$\widetilde{\mathrm{ev}}_V : \ V \otimes V^* \to \Bbbk, \qquad \text{given by} \quad v \otimes f \mapsto f(\boldsymbol{g} v),$$
$$\widetilde{\mathrm{coev}}_V : \ \Bbbk \to V^* \otimes V, \qquad \text{given by} \quad 1 \mapsto \sum_i v_i^* \otimes \boldsymbol{g}^{-1} v_i \ .$$

- Definition of right partial trace for $f \in \mathrm{End}_H(W \otimes V)$:

$$\mathrm{tr}^r_V(f) = (\mathrm{id}_W \otimes \widetilde{\mathrm{ev}}_V) \circ (f \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_W \otimes \mathrm{coev}_V) \ =$$



$\in \ \mathrm{End}_H(W)$

Definition of modified trace on H-pmod

- A *right modified trace* on H-pmod is a family of linear functions

$$\{\mathrm{t}_P \colon \mathrm{End}_H(P) \to \Bbbk\}_{P \in H\text{-pmod}}$$

satisfying cyclicity and right partial trace properties formulated below.

CYCLICITY If $P, Q \in H$-pmod, $f \colon P \to Q$, $g \colon Q \to P$, then

$$\mathrm{t}_P(g \circ f) = \mathrm{t}_Q(f \circ g)$$

RIGHT PARTIAL TRACE PROPERTY
If $P \in H$-pmod, $V \in H$-mod, $f \in \mathrm{End}_H(P \otimes V)$, then

$$\mathrm{t}_{P \otimes V}(f) = \mathrm{t}_P\big(\mathrm{tr}^r_V(f)\big)$$

- Similar for left.

Integral

- A *right integral* on H is a linear form $\boldsymbol{\mu}\colon H \to \mathbb{k}$ satisfying

$$(\boldsymbol{\mu} \otimes \mathrm{id})\Delta(x) = \boldsymbol{\mu}(x)\mathbf{1} \quad \text{for any} \quad x \in H.$$

- If H is finite-dimensional, the space of solutions of this equation is known to be 1-dimensional.
- The comodulus is the element $\boldsymbol{a} \in H$ defined with any non trivial right integral $\boldsymbol{\mu}$ by

$$(\mathrm{id} \otimes \boldsymbol{\mu})\Delta(x) = \boldsymbol{\mu}(x)\boldsymbol{a} \quad \text{for any} \quad x \in H.$$

Unimodular, unibalanced

- H is unimodular iff there exists a non trivial $\boldsymbol{c} \in H$ (2-sided cointegral) such that

$$x\boldsymbol{c} = \epsilon(x)\boldsymbol{c} = \boldsymbol{c}x \text{ for all } x \in H.$$

- $(H, \boldsymbol{g})$ is unibalanced iff $\boldsymbol{g}^2 = \boldsymbol{a}$.

Results

Let $(H, \boldsymbol{g})$ be a pivotal Hopf algebra.

- Theorem 1 [BBGa]
  There exists a non trivial right modified trace on H-pmod if and only if H is unimodular. It is unique up to scalar and non degenerate. A formula for the regular representation is as follows.

$$t_H(f) = \mu(\boldsymbol{g} f(1)) = \mu_{\boldsymbol{g}}(f(1)) \ .$$

  Here $\mu_{\boldsymbol{g}} = \mu(\boldsymbol{g}\ .\ )$. It is a non degenerate symmetric linear fonction on H.
- Theorem 2 [BBGa]
  There exists a non trivial 2-sided modified trace on H-pmod if and only if H is unimodular with unibalanced pivotal structure.

Modified trace pairings

- Proposition.
  $\mu_{\boldsymbol{g}}$ induces a non degenerate pairing

$$\begin{array}{rcl} Z(H) \otimes \mathrm{HH}_0(H) & \to & \Bbbk \\ x \otimes [y] & \mapsto & \mu_{\boldsymbol{g}}(xy) \end{array} .$$

  Here $Z(H)$ is the center, $\mathrm{HH}_0(H) = H/[H,H]$.
- On $H^{\otimes m}$ we have right action r_y for $y \in H^{\otimes m}$,
  and a left action l_x , for $x \in \left(H^{\otimes m}\right)^H$ (centraliser for left H-action).
  The modified trace defines a pairing

$$\begin{array}{rcl} \left(H^{\otimes m}\right)^H \otimes H^{\otimes m} & \to & \Bbbk \\ x \otimes y & \mapsto & \langle x, y \rangle = t_{H^{\otimes m}}(l_x \circ r_y) \end{array}$$

- The modified trace pairing induces a non degenerate pairing

$$\left(H^{\otimes m}\right)^H \otimes \mathrm{HH}_0(H, H^{\otimes m}) \to \Bbbk$$

  Here $\mathrm{HH}_0(H, H^{\otimes m}) = H^{\otimes m}/hx - xh$.

Universal invariant for a string link

- Here H is a finite dimensional ribbon Hopf algebra, which implies unimodular with unibalanced pivotal structure.
- A string link T is a tangle without closed component in which source points are connected to target points in the same order.
- Using a local receipe one associates to an m components string link T its universal univariant, $J_T \in \left(H^{\otimes m}\right)^H$.

Quantum characters and colored Hennings of links

- $\mathrm{qChar}(H) = \{f \in H^*,\ \forall x, y\ f(xy) = f(S^2(y)x)\}$.
- $\mathrm{qChar}(H)$ is a free module on the center $Z(H)$ with basis the right integral μ.
- Each object V defines a quantum character qTr_V but in non-semisimple case those do not generate $\mathrm{qChar}(H)$. There are so called pseudo-characters.
- By evaluating J_T with quantum characters one obtains invariants of the closure link $L = \widehat{T}$. This defines the Hennings invariant of an m-components framed link L, colored with center elements $x = (x_1, \dots, x_m)$.

$$H(L, x) = (\otimes_j x_j \mu)(J_T) \ .$$

Hopf link pairing

- In non-semisimple examples Hennings invariant with center colors for the Hopf link is degenerate on $Z(H)^{\otimes 2}$.
- If we use a center element x (or qCharacter $x\mu$) on one component, we get a center element $D(x)$ (Drinfeld map) which can be further evaluated against trace classes $h \in \mathrm{HH}_0(H)$. We get a modified Hopf link pairing

$$x \otimes h \mapsto \mu_{\mathbf{g}}(D(x)h) \ .$$

- In the factorizable case, the modified Hopf link pairing is non degenerate on $Z(H) \otimes \mathrm{HH}_0(H)$.

Logarithmic invariant for links

- New idea is to use both center elements and trace classes as colors.

Theorem (BBGe in case of restricted quantum $sl(2)$)

*Let (L^+, L^-) be a link with m_+ components in L^+ colored with central elements $x = (x_j)$, and m_- components in L^- colored with trace classes $y = (y_k)$.
Suppose L is the closure of the string link $T = (T^+, T^-)$ with universal invariant J_T.
Then*

$$X = \left((\otimes_j x_j \mu) \otimes \mathrm{id}\right)(J_T) \in \left(H^{\otimes m_-}\right)^H ,$$

and the following defines an invariant of the colored link $L = ((L^+, x), (L^-, y))$.

$$\mathrm{H}^{\log}(L) = < X, y > \quad .$$

- Extension to colored links in 3-manifolds by evaluating μ on surgery components.

TFT picture

- In non semisimple TFT, the natural map $V(-\Sigma) \to V(\Sigma)^*$ may not be an isomorphism.
- For the TFT V extending the above logarithmic invariant, we have

$$V(S^1 \times S^1) \cong HH_0(H)\ ,\ V(S^1 \times S^1)^* = V'(S^1 \times S^1)^* \cong Z(H)\ .$$

- The full TFT extension (strictly functorial and monoidal) was constructed by Geer-Patureau-De Renzi. The Kerler-Lyubashenko TFT vector spaces are recovered.