CORRELATORS FOR FINITE CONFORMAL FIELD THEORIES

Plan

Basic goal in CFT :

☞ determine all correlation functions (as in any quantum field theory)

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- in terms of basic algebraic / combinatorial data
- specifically: as morphisms in appropriate category

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Plan :

- *full* vs *chiral* CFT
- bulk field correlators for finite CFTs via a Lego game
- sewing constraints for open-closed CFTs
- example : boundary states and annulus amplitudes in the Cardy case

Full vs chiral CFT

Full vs chiral CFT

CFT

=

system of **correlators**

- ☞ solutions to Ward identities
- ☞ invariant under mapping class groups
- ☞ compatible with sewing / cutting & gluing

CFT

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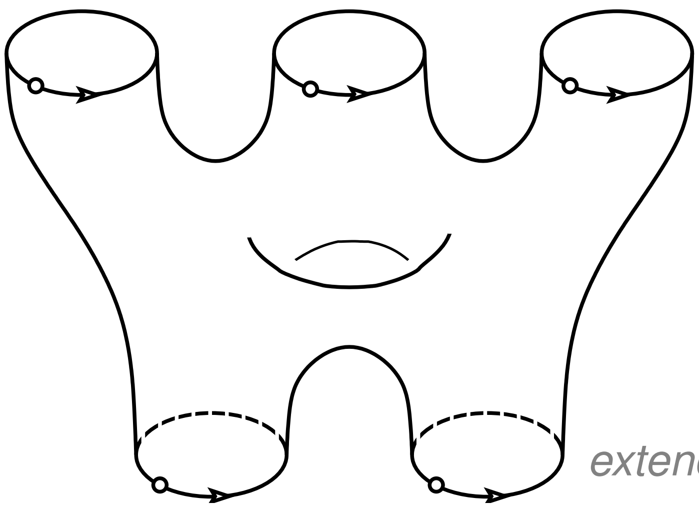

extended surface

Full vs chiral CFT

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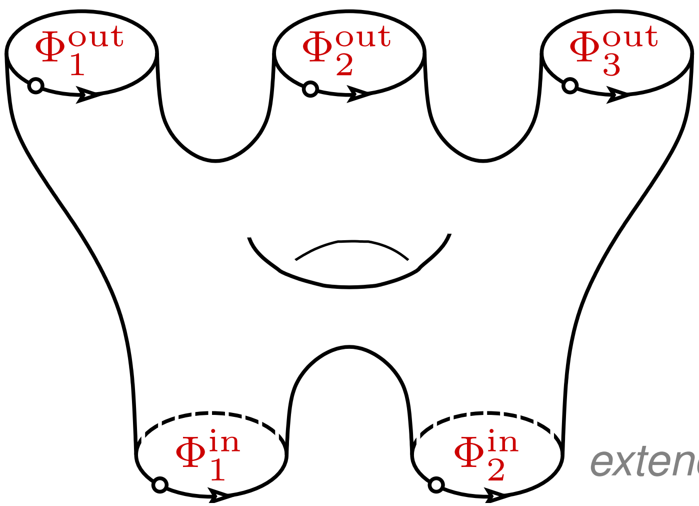

extended surface with field insertions

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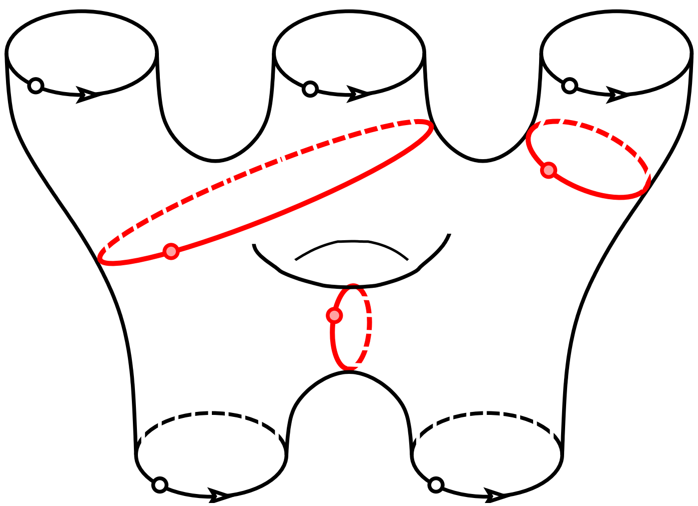

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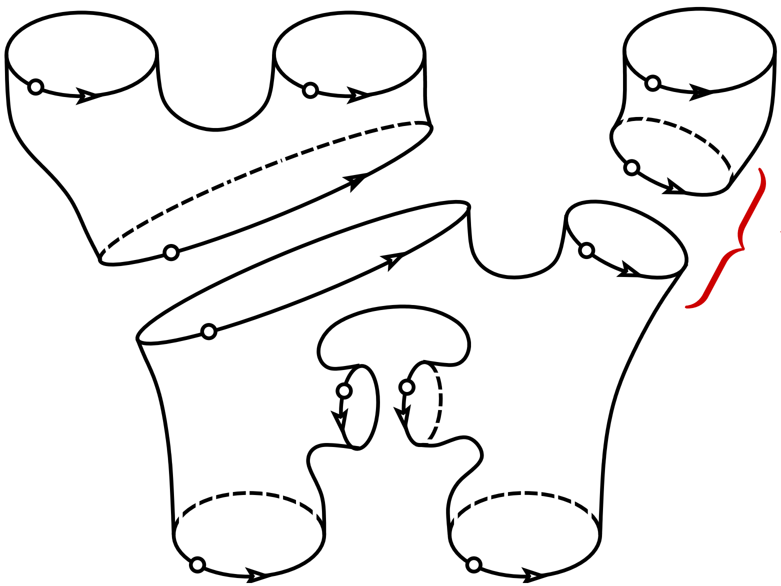

'sum over intermediate states'

Full vs chiral CFT

Full local CFT

$=$

system of **correlators**

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Full vs chiral CFT

Chiral CFT

=

system of spaces of **conformal blocks**

- ☞ solutions to chiral Ward identities
- ☞ carry actions of mapping class groups
- ☞ linear maps implementing sewing

Full local CFT

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system of **correlators**

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Full vs chiral CFT

Chiral CFT	**Full local CFT**
=	=
system of spaces of **conformal blocks**	system of **correlators**
☞ solutions to chiral Ward identities	☞ solutions to Ward identities
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☞ **Principle: Correlator = specific element in suitable space of conformal blocks** ☜

Full vs chiral CFT

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Full local CFT

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Principle: Correlator = specific element in suitable space of conformal blocks

- conformal blocks as vector spaces with actions of mapping class groups ...
- realized as morphism spaces of some braided monoidal category $\mathcal{D}$
- require locality $\rightsquigarrow$ take $\mathcal{D} \simeq \mathcal{Z}(\mathcal{C})$ for some braided monoidal category $\mathcal{C}$
- amenability $\rightsquigarrow$ take $\mathcal{C}$ modular : $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

Interlude: RCFT correlators

FRÖHLICH-FELDER-JF-SCHWEIGERT 2000

JF-RUNKEL-SCHWEIGERT 2002

· · · · · ·

FJELSTAD-JF-RUNKEL-SCHWEIGERT 2008

Rational CFT

- semisimple modular tensor category $\mathcal{C}$
  - can make use of chiral RCFT $\longleftrightarrow$ R-T TFT
  - can work with simple objects $x_i \boxtimes x_j$ in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

world sheet $X \longmapsto M_X$ 3-manifold with ribbon graph

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world sheet $\boldsymbol{X} \longmapsto \boldsymbol{M_X}$ 3-manifold with ribbon graph

$$\boldsymbol{M_X} = \left(\emptyset \xrightarrow[\boldsymbol{M_X}]{} \widehat{\boldsymbol{X}} \right)$$

$$\mathit{Cor}(\boldsymbol{X}) := \text{R-T}_{\mathcal{C}}(\boldsymbol{M_X})\,.\,\mathbf{1} \;\in\; \text{R-T}_{\mathcal{C}}(\widehat{\boldsymbol{X}}) \;=\; \text{blocks for } X$$

$\widehat{X}$ = oriented double of world sheet X

recall : $\text{R-T}_{\mathcal{C}}(M_X) : \mathbb{C} = \text{R-T}_{\mathcal{C}}(\emptyset) \longrightarrow \text{R-T}_{\mathcal{C}}(\widehat{X})$

thus $\text{R-T}_{\mathcal{C}}(M_X)\,.\,1$ vector in conformal block space $\text{R-T}_{\mathcal{C}}(\widehat{X})$

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arbitrary X — possibly w/ boundary / w/ topological defects
/ possibly unoriented
/ arbitrary field insertions

Main RCFT results

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THEOREM — **Classification**

there is a bijection

non-degenerate full RCFTs with chiral theory based on m.t.c. $\mathcal{C}$

$\longleftrightarrow$ Morita classes of ♒♒♒ Frobenius algebras in $\mathcal{C}$

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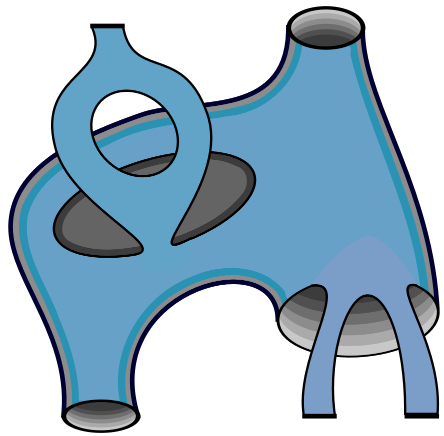
simple special symmetric
for correlators on oriented world sheets

simple special symmetric and with a *Jandl* structure
for correlators on oriented and unoriented world sheets



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KONG-RUNKEL 2009

DAVYDOV-MÜGER-NIKSHYCH-OSTRIK 2013

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THEOREM — **Construction of correlators**

construction of correlator *Cor*(X)

$=$ prescription for ribbon graph in connecting manifold M_X :

… … … … …

Beyond rationality

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☞ convenient characterization of modularity :

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left and right chiral halves

full theory

Beyond rationality

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  makes sense for any finite ribbon category

  $\Longrightarrow$ natural to waive semisimplicity while keeping finiteness : **finite CFT**

- ☞ *Challenge* : no associated R-T 3-d TFT
- ☞ *Problem* : objects for bulk fields not just direct sums of factorized objects

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☞ but :

- still spaces of conformal blocks as morphism spaces
  carrying rep's of mapping class groups and satisfying sewing relations

LYUBASHENKO 1995

- still can play Lego game

Finite CFTs: Lego-Teichmüller game

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Finite CFTs : based on possibly non-semisimple modular tensor category $\mathcal{D}$

- ☞ for now restrict attention to bulk theory / do not assume $\mathcal{D}$ to be a center
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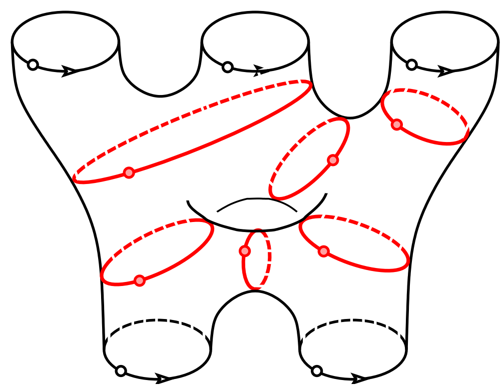
- pair-of-pants decomposition
- spheres with at most three holes / pairs-of-pants as building blocks

GROTHENDIECK 1984

HATCHER-THURSTON 1980

HARER 1983

BAKALOV-KIRILLOV 2000



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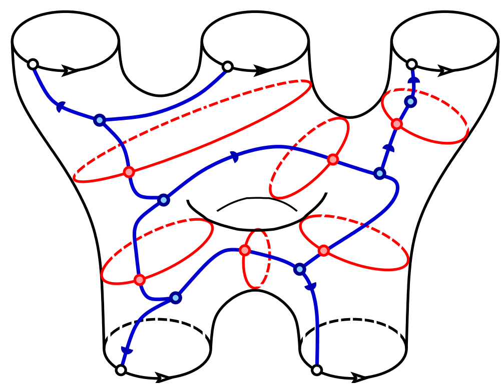
  marking : embedded graph with a vertex on each boundary circle



marked surface

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☞ play Lego-Teichmüller game

- pair-of-pants decomposition
- spheres with at most three holes / pairs-of-pants as building blocks
- keep track of order of sewings by auxiliary structure :
  marking: embedded graph with a vertex on each boundary circle
- relate markings by sequence of elementary moves
  (Z-move, B-move, F-move, A-move, S-move)
- smallish number of constraints among sequences of elementary moves
- defines groupoid of marked surfaces presented by generators and relations
- groupoid is connected and simply connected

Finite CFTs: Lego-Teichmüller game . . .

☞ Lego-Teichmüller $\Longrightarrow$ control over pair-of-pants decompositions / sewings

☞ Problem : markings as auxiliary data – correlators must not depend on them

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Solution :

JF-Schweigert 2017

- take whole *bulk object* F at each insertion
- interpret system of correlators as monoidal natural transformation v_F from constant functor $\Delta_{\Bbbk}$ to block functor $\mathrm{Bl}^{(F)}$
- first work with marked surfaces
  $\rightsquigarrow$ *pre*-correlators $\widetilde{\mathrm{v}}_F : \widetilde{\Delta}_{\Bbbk} \Longrightarrow \widetilde{\mathrm{Bl}}^{(F)}$

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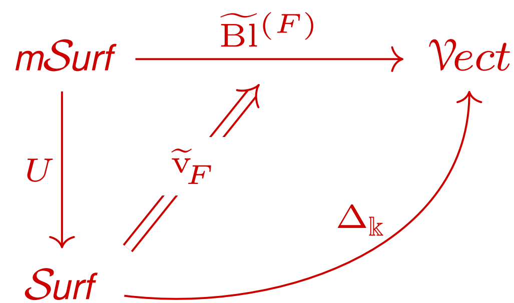
- find pre-correlators
- to get rid of dependence on auxiliary data consider right Kan extension along forget functor



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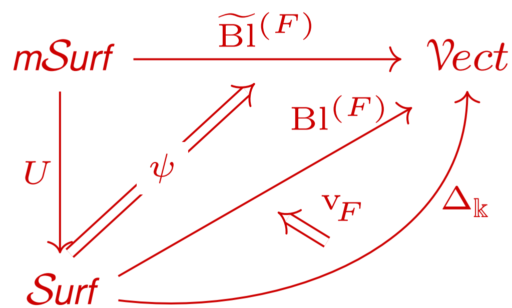

($\mathrm{Bl}^{(F)}$ exists and has natural symmetric monoidal structure)

Chiral finite CFT: Block functors

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Chiral CFT $\rightsquigarrow$ system of vector spaces of conformal blocks :

☞ for each extended surface $X = X^{g}_{p|q}$ a functor

$$\mathrm{Bl}^{g}_{p|q} : \quad \mathcal{D}^{\boxtimes(p+q)} \longrightarrow \mathcal{V}ect$$

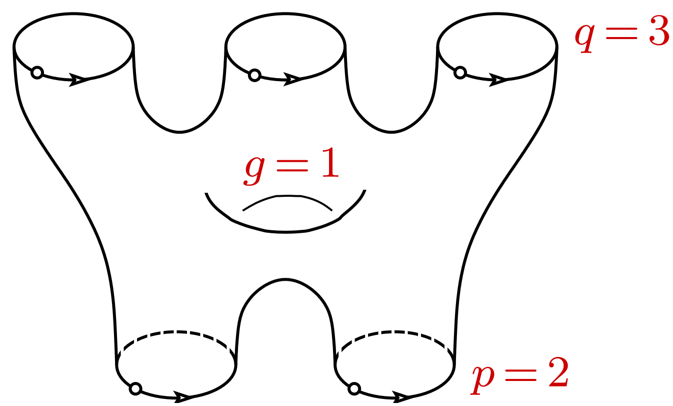
genus

number of incoming / outgoing insertions

contravariant / covariant labels (objects of $\mathcal{D}$)

such that

↯ . . . . . .



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- mapping class group Map$(X^g_{p|q})$ acting projectively on the spaces $\{\mathrm{Bl}^g_{p|q}(-)\}$
- functorial linear maps $\mathrm{Bl}_1(-;x) \otimes_{\Bbbk} \mathrm{Bl}_2(x^\vee;-) \longrightarrow \mathrm{Bl}_\#(--)$
  with suitable properties
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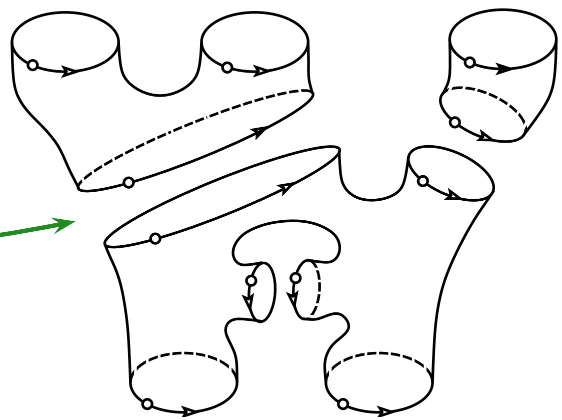

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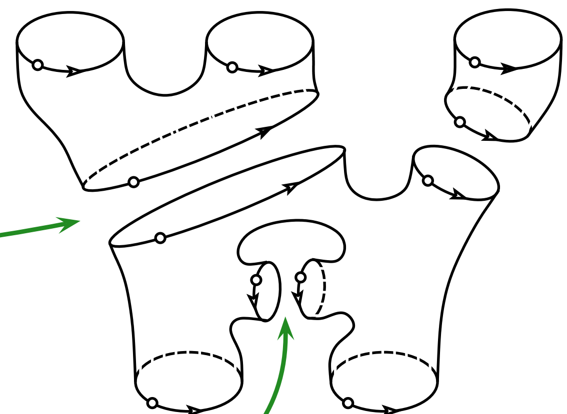
  for any sewing $(X_1, X_2) \longrightarrow X_{\#} = X_1 \# X_2$
- analogous maps
  for self-sewings



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THEOREM — **Construction of blocks**

associated with any modular finite ribbon category $\mathcal{D}$

there is a system of conformal blocks with values in morphism spaces :

$$\mathrm{Bl}^g_{p|q}(u_1, \ldots, u_{p+q}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, u_1^\wedge \otimes u_2^\wedge \otimes \cdots \otimes u_{p+q}^\wedge \otimes K^{\otimes g})$$

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- $K = \int^{x \in \mathcal{D}} x \otimes {}^{\vee}x$ $\qquad u^{\wedge} = \begin{cases} u^{\vee} & \text{if incoming} \\ u & \text{if outgoing} \end{cases}$
- K has a natural structure of Hopf algebra with Hopf pairing
- $\mathcal{D}$ modular $\Longleftrightarrow$ Hopf pairing non-degenerate

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- construction via sewing genus-0 three-point blocks
  using only structural data of $\mathcal{D}$ — including structure on K

Sewing conformal blocks

sewing is *local* operation but still distinguish two types of sewing :

☞ *non-handle creating* :

⇝ amounts to coend $\int^{x\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(u,x) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{D}}(x,v) \;=\; \mathrm{Hom}_{\mathcal{D}}(u,v)$

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  - to be regarded as coend in category of left exact functors ($u,\, v$ as variables)
  - imposing universality only w r t left exact functors ensures representability
  - then $\int^{x\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(u, v\otimes x\otimes x^{\vee}) = \mathrm{Hom}_{\mathcal{D}}(u, v\otimes K)$

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Blocks for marked surfaces

Blocks for marked surfaces

chiral CFT $\rightsquigarrow$ system of vector spaces of conformal blocks :

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depends on
auxiliary data :
marked surfaces

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☞ assign abstract vector space $\mathrm{Bl}^g_{p|q}(-)$ to extended surface $X^g_{p|q}$

☞ assign concrete vector space $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},-)$ to $X^g_{p|q}$ + *auxiliary gluing data* Γ

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☞ Task : get control over isomorphisms between blocks for different auxiliary data

☞ Tool : Lego-Teichmüller game

$\Longrightarrow$ unique isomorphism between vector spaces for any two gluing data

Tool : coends

$\Longrightarrow$ unique isomorphisms canonically specified via Fubini theorems

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chiral CFT $\rightsquigarrow$ system of vector spaces of conformal blocks :

$$\mathrm{Bl}^{g}_{p|q}(u_1,\dots,u_{p+q}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, u_1^{\wedge}\otimes u_2^{\wedge}\otimes\cdots\otimes u_{p+q}^{\wedge}\otimes K^{\otimes g})$$

- assign abstract vector space $\mathrm{Bl}^{g}_{p|q}(-)$ to extended surface $X^{g}_{p|q}$
- assign concrete vector space $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},-)$ to $X^{g}_{p|q}$ + *auxiliary gluing data* Γ
- Task : get control over isomorphisms between blocks for different auxiliary data
- Tool : Lego-Teichmüller game

  $\Longrightarrow$ unique isomorphism between vector spaces for any two gluing data

  Tool : coends

  $\Longrightarrow$ unique isomorphisms canonically specified via Fubini theorems
- represent generators by linear maps on conformal block spaces :

  Z-isomorphism ...... S-isomorphism

  Result : can be done in such a way that all relations satisfied

  modulo central extension of genus-1 relation $(\mathrm{S}\circ\mathrm{T}_c)^3 = \mathrm{C}\circ\mathrm{S}^2$

Full finite bulk CFT: Main results

Results :

JF-SCHWEIGERT 2017

Full finite bulk CFT: Main results

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- ☞ Construction : starting with candidate bulk object F and any three morphisms

  $$\varepsilon_F \in \mathrm{Hom}_{\mathcal{D}}(F, \mathbf{1}) \qquad \Phi_F \in \mathrm{Hom}_{\mathcal{D}}(F, F^{\vee}) \qquad \omega_F \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F^{\otimes 3})$$

  obtain candidate vector $\widetilde{\mathrm{v}}_F(X, \Gamma)$ in each space $\widetilde{\mathrm{Bl}}^{(F)}(X, \Gamma)$ via sewing

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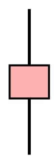
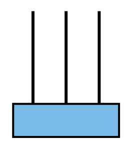
graphically : $\varepsilon_F =$  $\Phi_F =$  $\omega_F =$ 

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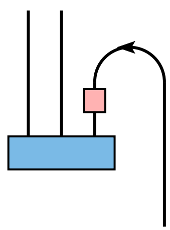
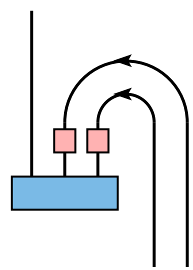
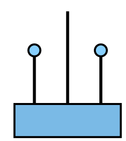
  $\Longrightarrow$ F has natural structure of Frobenius algebra $(F, m_F, \eta_F, \Delta_F, \varepsilon_F)$

$\Delta_F :=$  $\in \mathrm{Hom}_{\mathcal{D}}(F,F\otimes F)$

$m_F :=$  $\in \mathrm{Hom}_{\mathcal{D}}(F\otimes F,F)$

$\eta_F :=$  $\in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1},F)$

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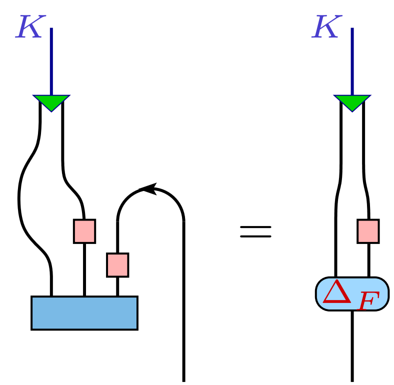
☞ Definition : *modular* Frobenius algebra

$:\Longleftrightarrow$ commutative symmetric and

S-invariant : $S_K \circ \overline{\mathcal{T}}_F = \overline{\mathcal{T}}_F$

$\overline{\mathcal{T}}_F :=$ 

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THEOREM — **Classification**

for $\mathcal{D}$ ~~modular~~ finite ribbon category there is a bijection

non-degenerate *genus-0* monoidal natural transformations $\tilde{\mathrm{v}}_F$

$\longleftrightarrow$ isomorphism classes of

commutative symmetric Frobenius algebras $F \in \mathcal{D}$

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CONJECTURE — **Existence**

for any modular tensor category $\mathcal{C}$ and any ribbon automorphism ω of $\mathcal{C}$

the object $F_\omega := \int^{x\in\mathcal{C}} \omega(x) \boxtimes {}^\vee x \in \mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}$

carries a natural structure of modular Frobenius algebra

☞ in particular : **non-semisimple full finite CFTs exist**

☞ $\omega = \mathrm{Id}$: "*Cardy case*"

Full finite bulk CFT: Main results

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THEOREM — **Existence**

for any modular Hopf $\Bbbk$-algebra H and any ribbon automorphism ω of H

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JF-SCHWEIGERT-STIGNER 2012

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correlator of a full finite CFT for closed surface of genus g

with p incoming and q outgoing bulk fields :

$$\widetilde{\mathrm{v}}_F(X^g_{p|q}, \Gamma_\circ) = \Delta_F^{(q-1)} \circ \mathcal{T}_F^{(g)} \circ m_F^{(p-1)}$$

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defuse outgoing — add handles — fuse incoming

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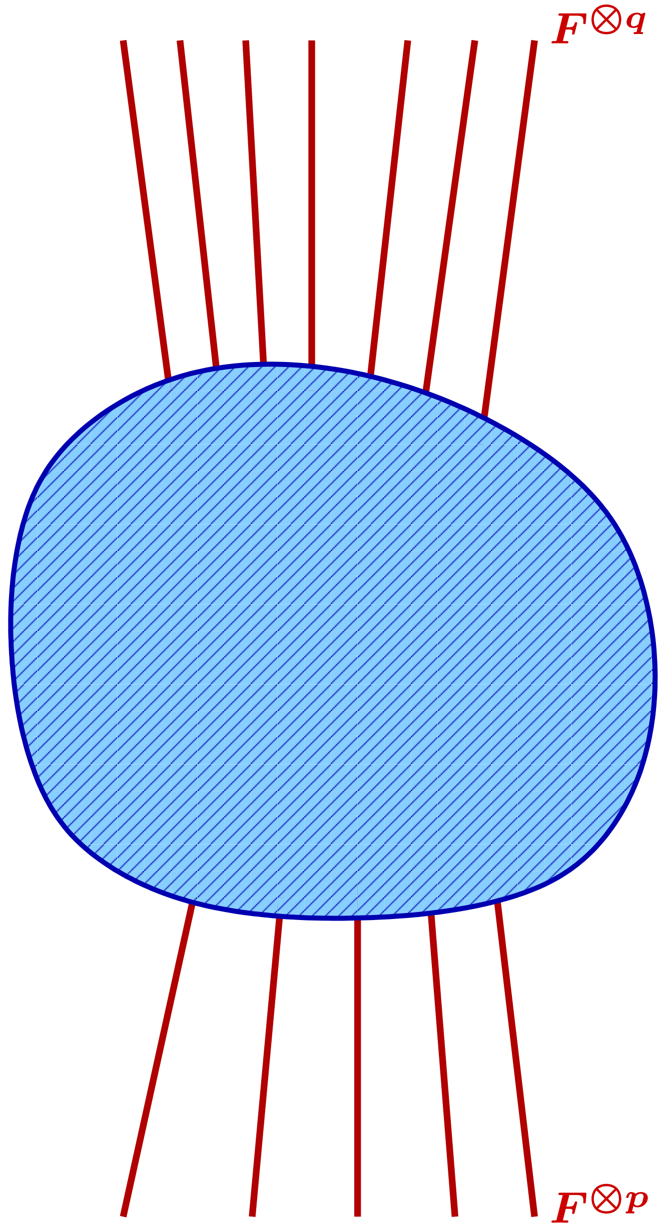

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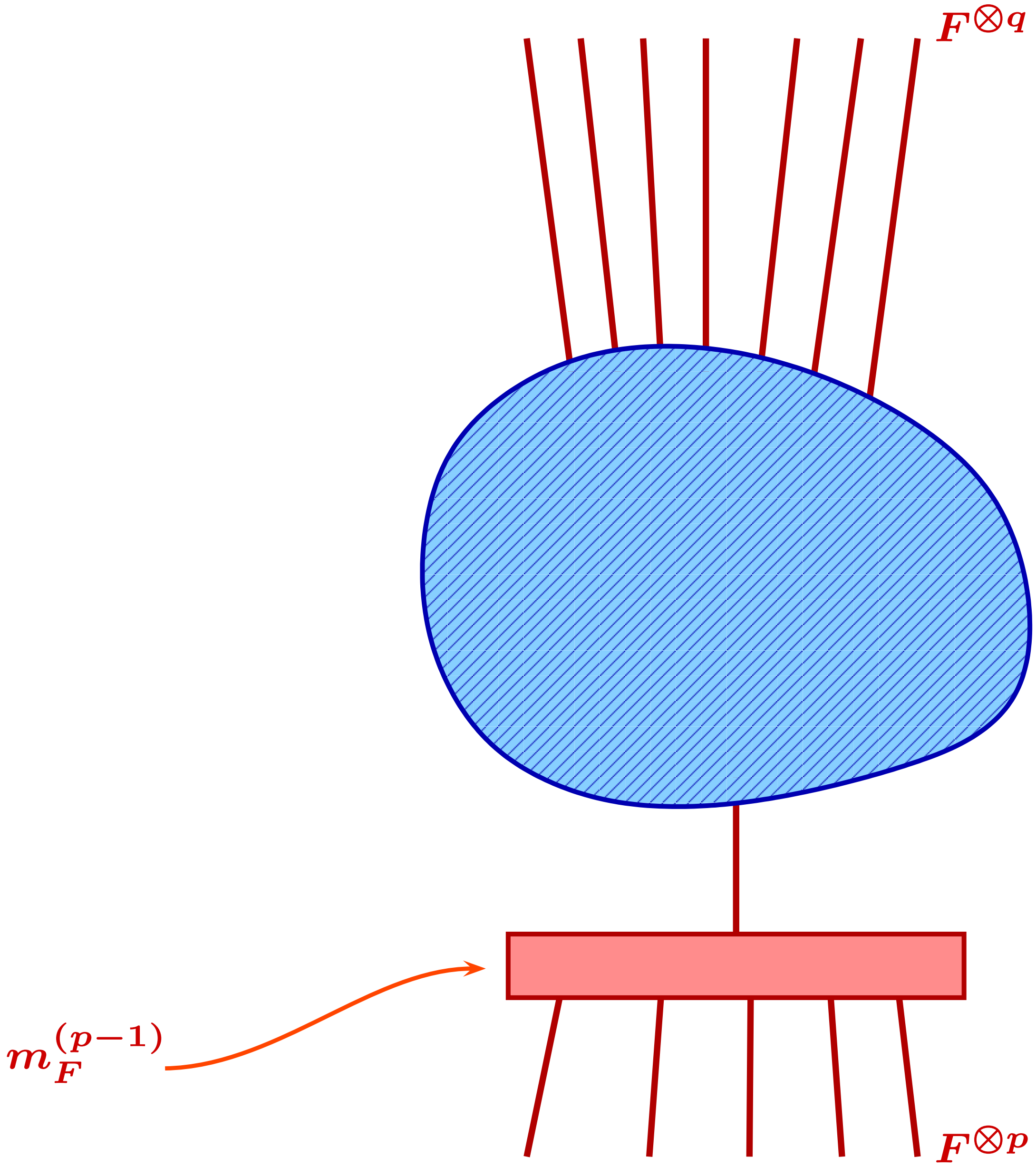

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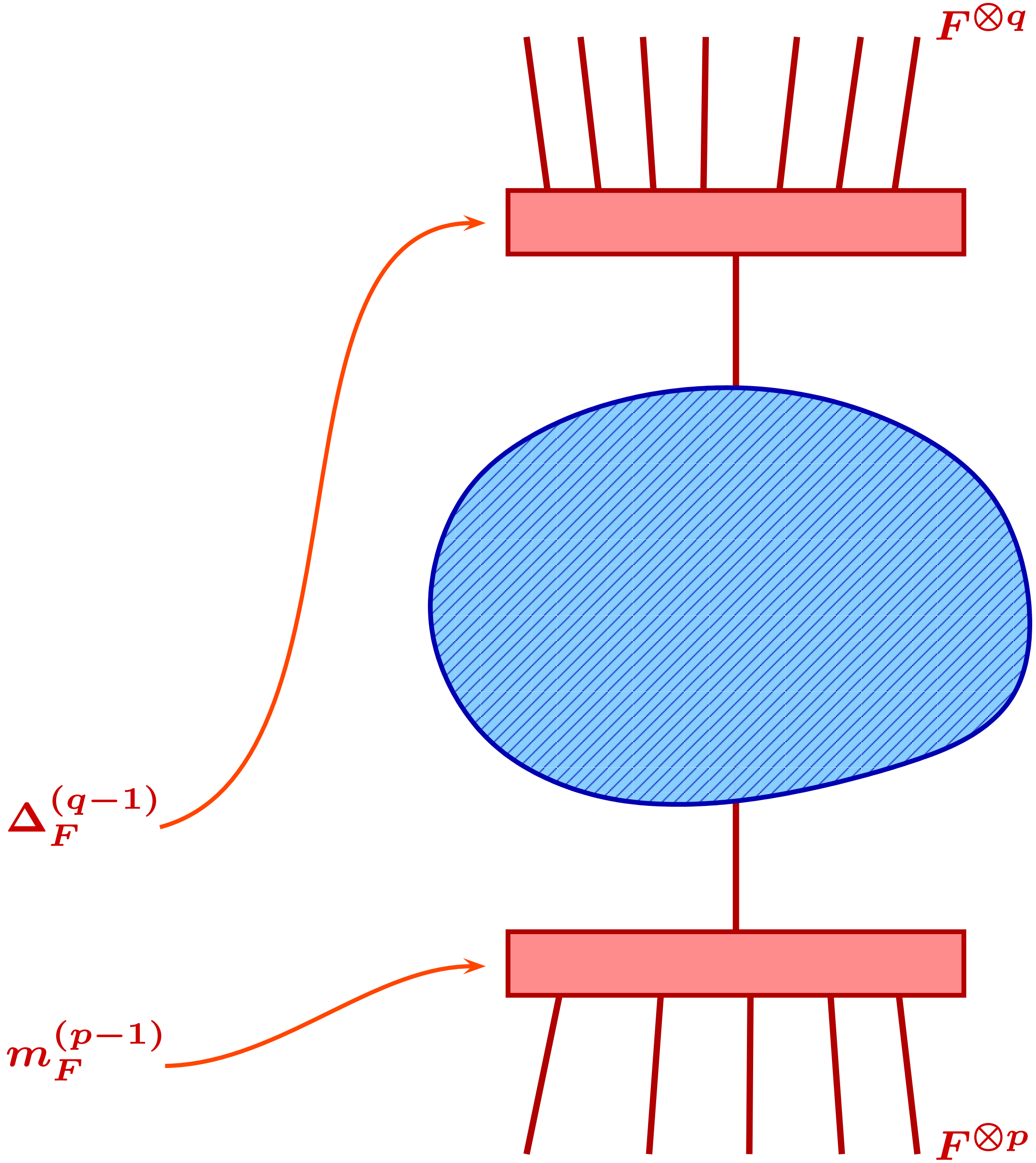

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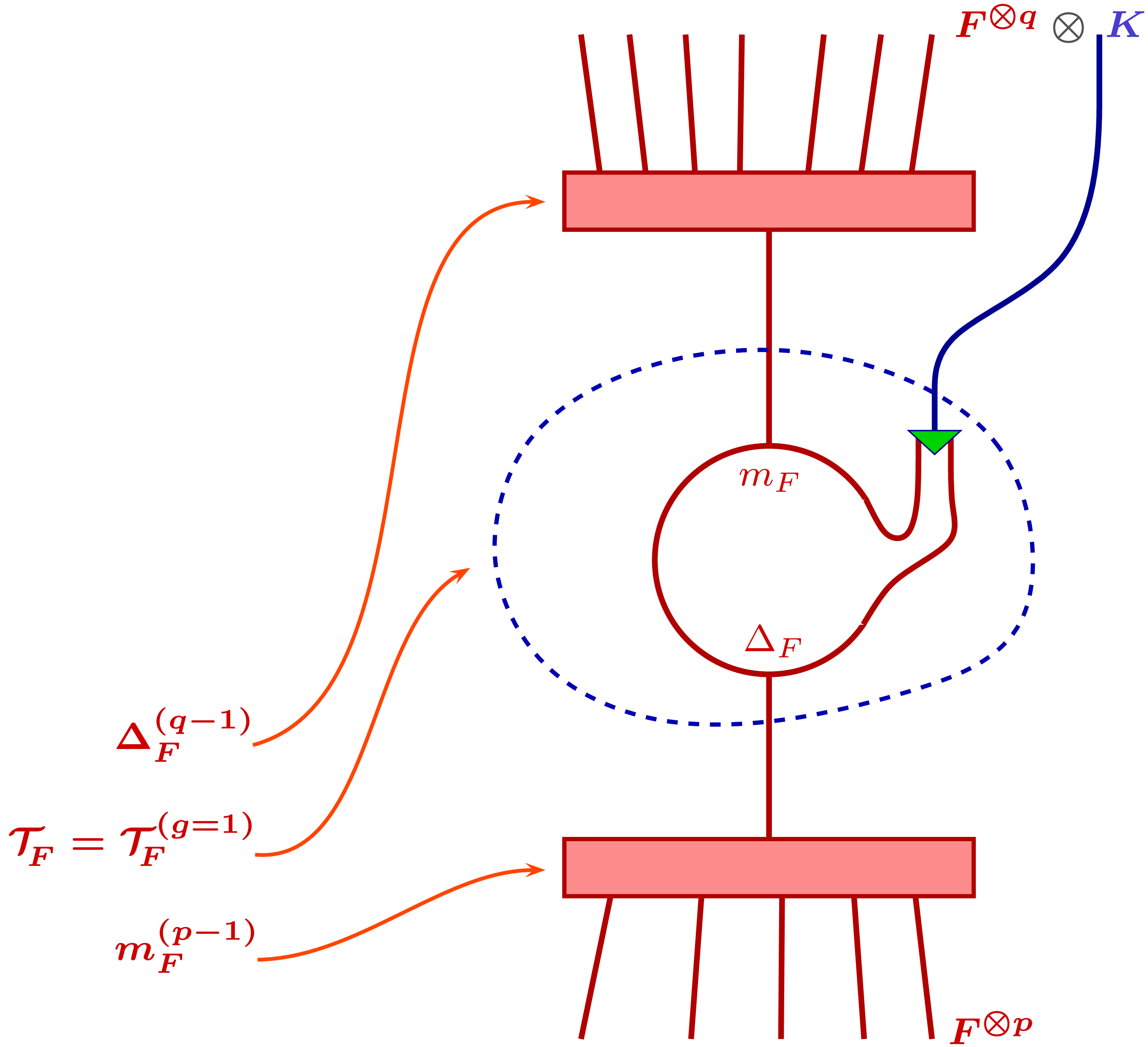

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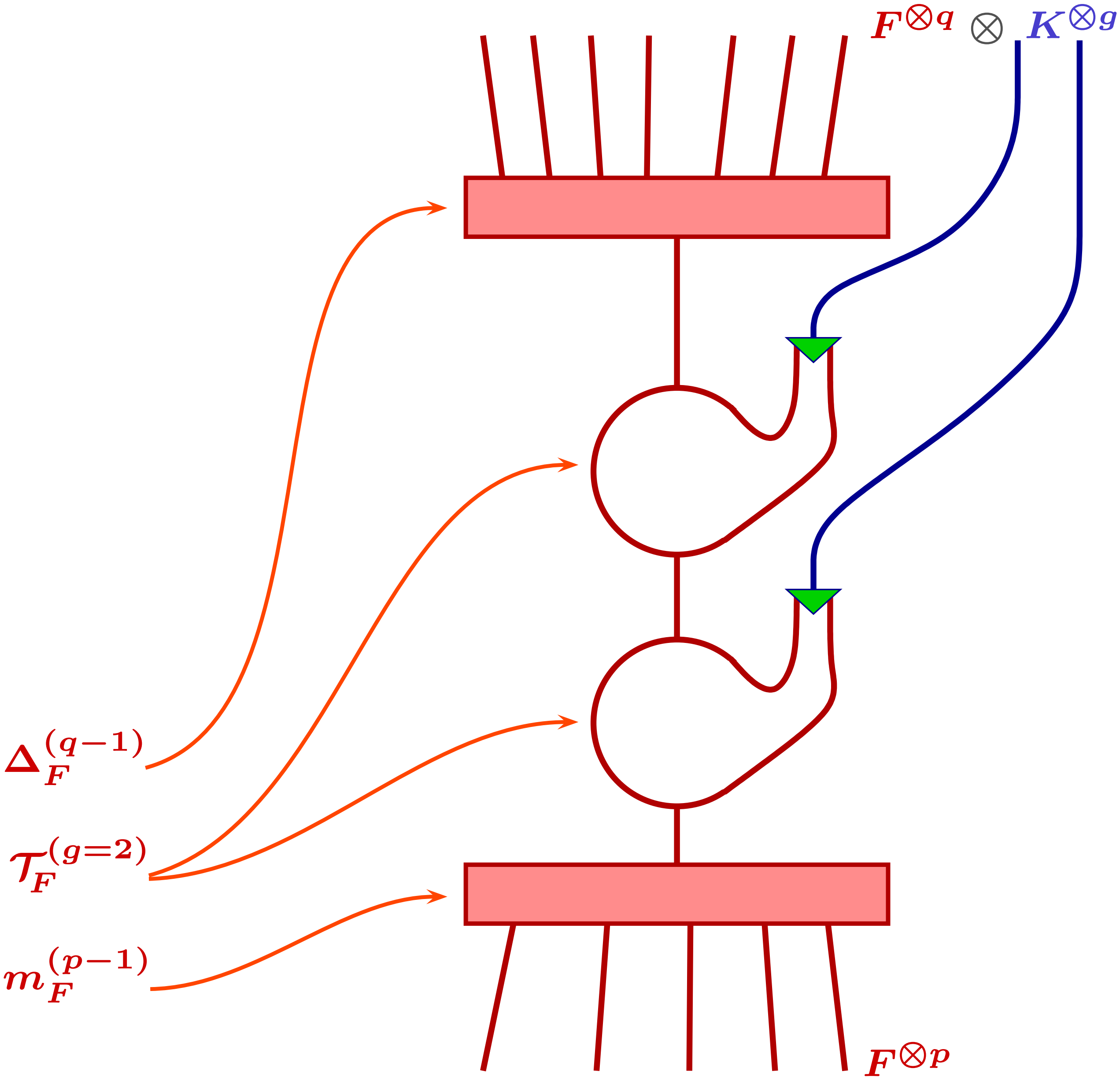

Surfaces with boundary

JF-Gannon-Schaumann-Schweigert 2018 / in prep.

Surfaces with boundary

Next step : allow for surfaces with boundary

(labeled free boundaries in addition to gluing boundaries)

but still oriented and without defects

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  - 1 bulk and 1 boundary field on a disk $\in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, U_{\mathcal{Z}}(F) \otimes B_{mm})$

when all insertions incoming

- bulk state space F and
  boundary state spaces B_{mn} changing boundary condition from m to n
- sewing via coends

Surfaces with boundary

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- endofunctor $Z\colon\; c \longmapsto \int^{x\in\mathcal{C}} x \otimes c \otimes {}^{\vee}x$
- dinatural family for the coend endows $Z(c)$ with half-braiding
- indeed $\mathcal{Z}(\mathcal{C}) \simeq Z$-mod
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$\Longrightarrow$ can switch back and forth between $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$

Lewellen's constraints

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☞ **crossing symmetry on the sphere & modular S-invariance**

- ⇝ Duplo version of purely closed theory
- ⇝ bulk object F modular Frobenius algebra in $\mathcal{Z}(\mathcal{C})$

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can be analyzed similarly as crossing symmetry on the sphere

⇝ boundary objects $\{B_{mn}\}$ form symmetric Frobenius algebroid in $\mathcal{C}$

➻ with m, n objects of category $\mathcal{M}$ of boundary conditions

➻ expect : $\mathcal{M}$ an exact $\mathcal{C}$-module

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- **genus-0 bulk-boundary compatibility** :
  - comparison of two boundary factorizations

    for correlator of 1 bulk and 2 boundary fields on a disk

    $\rightsquigarrow$ presence of natural Z-module structure on each B_{mn}
  - comparison of a bulk and a boundary factorization

    for correlator of 1 boundary and 2 bulk fields on a disk

    $\rightsquigarrow$ multiplication morphism $Z \circ Z(B_m) \to Z(B_m)$

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comparison of a bulk and a boundary factorization

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☞ constraints can be directly (and almost are) verified in Cardy case :

- $\mathcal{M} = \mathcal{C}$ regular $\mathcal{C}$-module
- $F = \int^{x \in \mathcal{C}} x \otimes {}^{\vee}x \ \in \mathcal{Z}(\mathcal{C})$
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- $B_{mn} = m^{\vee} \otimes n = \underline{\mathrm{Hom}}_{\mathcal{C}}(m,n) \;\in \mathcal{C}$
- also e.g. : torus partition function $Z = \sum_{i,j} C_{ij}\, \chi_i \otimes \overline{\chi}_{j^{\vee}}$
  Cardy-Cartan modular invariant

☞ Side remark : conjectural generalization :

- $B_{mn} = \underline{\mathrm{Hom}}_{\mathcal{M}}(m,n)$
- F as ~~co~~end over $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,n)$ (inner natural transformations of $\mathrm{Id}_{\mathcal{M}}$)

Boundary states

Boundary states

RESULT — **Boundary states & annuli**

for any finite modular tensor category $\mathcal{C}$ in the Cardy case one has :

- *boundary states* $=$ (co)characters of L-modules

  with $L = \int^{x\in\mathcal{C}} x \otimes {}^{\vee}x \;\equiv\; U_{\mathcal{Z}}(F) \;\in \mathcal{C}$

- sewing these boundary states gives *annulus amplitudes*
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$=$ 0-point correlators on annulus

$=$ 1-point bulk field correlators on disk

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one lesson from considering annulus amplitudes :

- coend $\int^{x\in\mathcal{C}} x \otimes {}^\vee x$ is
  - Hopf algebra in $\mathcal{C}$ with Hopf pairing and integral
  - thereby also Frobenius algebra in $\mathcal{C}$
  - commutative Frobenius algebra in $\mathcal{Z}(\mathcal{C})$
- both Hopf pairing ω and Frobenius form κ such that $\omega \circ \kappa^{-1} = S_L$
- to control appearance of ω vs κ make use of adjunctions for the central monad


THANK YOU


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