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p-adic L-functions for $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$ II

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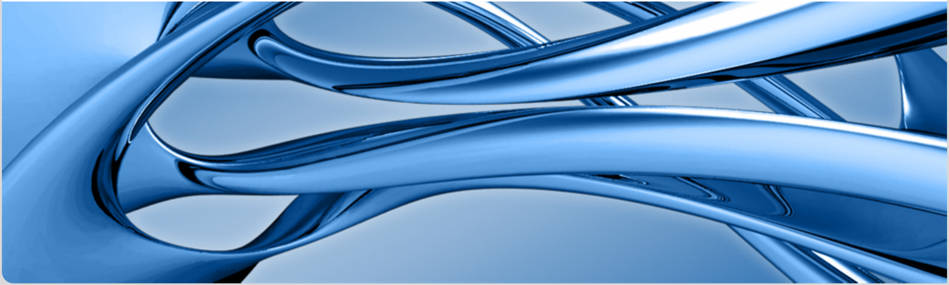

Overview

Plan of the lectures:

- Review of the analytic theory of Rankin-Selberg L-functions following Jacquet, Piatetski-Shapiro and Shalika
- The relative modular symbol and algebraicity of special values (Kazhdan-Mazur-Schmidt, Kasten-Schmidt, Raghuram-Shahidi, J., Raghuram)
- Archimedean periods: Non-vanishing and period relations (Kasten-Schmidt, Sun, J., Grobner-Lin)
- p-adic distributions attached to finite slope classes (Kazhdan-Mazur-Schmidt, Schmidt, J.)
- Boundedness in the nearly ordinary case (Schmidt, J.)
- Functional equation (J.)
- Manin congruences and independence of weight (J.)
- Interpolation formulae (Schmidt, J.)

Main References

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- **Analytic Theory**
  J. Cogdell, *Analytic theory of L-functions for* $\mathrm{GL}(n)$.
  Proc. 'School on Automorphic Forms, L-functions, and Number Theory'
  Institute for Advanced Studies Hebrew University, Jerusalem, March 12–16, 2001.
- **Algebraicity of L-values**
  A. Raghuram, *Critical values for Rankin-Selberg L-functions for* $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ *and the symmetric cube L-functions for* $\mathrm{GL}(2)$. Forum Math. **28** (2016) No. 3, 457-489.
- **(Archimedean) Periods**
  B. Sun, *The non-vanishing hypothesis at infinity for Rankin-Selberg convolutions.*
  Journal of the American Mathematical Society **30** (2017), 1–25.
  F. Januszewski, *On Period Relations for Automorphic L-functions I.*
  Transactions of the American Mathematical Society, in press.
  H. Grobner, J. Lin, *Special values of L-functions and the refined Gan-Gross-Prasad Conjecture*, Preprint.
- **Construction of p-adic L-functions**
  F. Januszewski, *Non-abelian p-adic Rankin-Selberg L-functions and non-vanishing of central L-values.* `http://arxiv.org/abs/1708.02616`

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Automorphic Representations

$F/\mathbf{Q}$ number field, $\mathbf{A}$ ring of over $\mathbf{Q}$, $\mathbf{A}_F = \mathbf{A} \otimes_{\mathbf{Q}} F$
Π irred. cusp. autom. rep. of $\mathrm{GL}_n(\mathbf{A}_F)$
$\mathrm{GL}_n \supset B_n = T_n U_n$ standard upper Borel subgroup
$\mathbf{1} \neq \psi : F\backslash\mathbf{A}_F \to \mathbf{C}^\times$, extends to $U_n(\mathbf{A}_F)$ via

$$(u_{ij}) \mapsto \prod_{i=1}^{n-1} \psi(u_{i,i+1}).$$

For any cusp for $\varphi \in \Pi \subseteq L^2(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F))$ we have its Fourier transform

$$\left[W_\varphi : \quad g \mapsto \int_{U_n(F)\backslash U_n(\mathbf{A}_F)} \varphi(ug)\psi^{-1}(u)du\right] \quad \in \quad \mathrm{Ind}_{U_n(\mathbf{A}_F)}^{\mathrm{GL}_n(\mathbf{A}_F)} \psi.$$

Shalika: $\mathscr{W}(\Pi, \psi) := \{W_\varphi \mid \varphi \in \Pi\}$ is non-zero, hence isomorphic to Π, since

$$\varphi(g) = \sum_{\gamma \in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\varphi(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot g).$$

Rankin-Selberg ζ-integrals

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$n \geq 1$ for the rest of the lectures
Π, Σ irred. cusp. autom. rep. of $\mathrm{GL}_{n+1}(\mathbf{A}_F)$ and $\mathrm{GL}_n(\mathbf{A}_F)$
Jacquet, Piatetski-Shapiro and Shalika consider for $\varphi \in \Pi$, $\varphi' \in \Sigma$:

$$I(s;\varphi,\varphi') := \int\limits_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F)} \varphi\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\right)\varphi'(g)|\det g|^{s-\frac{1}{2}}dg$$

Then:

- $I(s;\varphi,\varphi')$ converges absolutely and uniformly on compact sets for all $s \in \mathbf{C}$
- $I(s;\varphi,\varphi')$ defines an entire function in s, bounded in vertical strips
- With $\widetilde{\varphi}(g) := \varphi({}^tg^{-1})$ it satisfies the functional equation

$$I(s;\varphi,\varphi') = I(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

- $I(s;\varphi,\varphi')$ is Eulerian if $W_\varphi \in \mathscr{W}(\Pi,\psi)$ and $W_{\varphi'} \in \mathscr{W}(\Sigma,\psi^{-1})$ are factorizable
- For $n=1$ this is the ζ-integral of Jacquet-Langlands for $\mathrm{GL}(2)$

Rankin-Selberg ζ-integrals

Assume $W_\varphi = \otimes_v W_v$ and $W_{\varphi'} = \otimes_v W'_v$ with

$$W_v \in \mathscr{W}(\Pi_v, \psi_v), \quad W'_v \in \mathscr{W}(\Sigma_v, \psi_v^{-1})$$

Then for $\Re(s) \gg 0$:

$$\begin{aligned} I(s;\varphi,\varphi') &= \int\limits_{U_n(\mathbf{A}_F)\backslash \mathrm{GL}_n(\mathbf{A}_F)} W_\varphi\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right) W_{\varphi'}(g) |\det g|^{s-\frac{1}{2}} dg \\ &= \prod_v \int\limits_{U_n(F_v)\backslash \mathrm{GL}_n(F_v)} W_v\left(\begin{pmatrix} g_v & \\ & 1 \end{pmatrix}\right) W'_v(g_v) |\det g_v|^{s-\frac{1}{2}} dg_v \\ &= \prod_v \Psi(s; W_v, W'_v) \end{aligned}$$

Local L-functions: Non-archimedean case

Assume $v \nmid \infty$ and write q for the residue field cardinality.
For varying $W_v \in \mathscr{W}(\Pi_v, \psi_v)$ and $W'_v \in \mathscr{W}(\Sigma_v, \psi_v^{-1})$ the ζ-integrals

$$\Psi(s; W_v, W'_v) = \int\limits_{U_n(F_v)\backslash \mathrm{GL}_n(F_v)} W_v(\begin{pmatrix} g_v & \\ & 1 \end{pmatrix}) W'_v(g_v) |\det g_v|^{s-\frac{1}{2}} dg_v$$

span a fractional $\mathbf{C}[q^s, q^{-s}]$-ideal in $\mathbf{C}(q^{-s})$ containing 1.

Definition (Non-archimedean local L-functions)

Define $L(s, \Pi_v \times \Sigma_v)$ as the unique generator of the form

$$L(s, \Pi_v \times \Sigma_v) = \frac{1}{P(q^{-s})}, \quad P(X) \in \mathbf{C}[X], \; P(0) = 1.$$

Local L-functions: Non-archimedean case

Spherical case: If Π_v and Σ_v are *spherical*, Shintani's explicit formulae show

$$L(s, \Pi_v \times \Sigma_v) \;=\; \Psi(s; W_v^0, {W_v'}^0) \;=\; \frac{1}{\det(\mathbf{1} - q^{-s} A \otimes A')}$$

where $W_v^0 \in \mathscr{W}(\Pi_v, \psi_v)^{\mathrm{GL}_{n+1}(\mathcal{O})}$, ${W_v'}^0 \in \mathscr{W}(\Sigma_v, \psi_v^{-1})^{\mathrm{GL}_n(\mathcal{O})}$ are the spherical vectors and

$$A \;\in\; \mathrm{GL}_{n+1}(\mathbf{C}), \quad A' \;\in\; \mathrm{GL}_n(\mathbf{C})$$

are the Satake parameters for Π_v and Σ_v.

General case: $L(s, \Pi_v \times \Sigma_v)$ is always in the image of the map

$$\mathscr{W}(\Pi_v, \psi_v) \otimes \mathscr{W}(\Sigma_v, \psi_v^{-1}) \to \mathbf{C}(q^{-s}),$$

$$W_v \otimes W_v' \;\mapsto\; \Psi(s; W_v, W_v'),$$

i.e. it is a *finite linear combination of local ζ-integrals*.
Caution: The essential vectors do **not** yield $L(s, \Pi_v \times \Sigma_v)$ in general.

Local L-functions: Archimedean case

Assume $v \mid \infty$ and write $K_v \subseteq \mathrm{GL}_{n+1}(F_v)$ and $K'_v \subseteq \mathrm{GL}_n(F_v)$ for the (standard) maximal compact subgroups.

$$\Pi_v \quad \supset \quad \Pi_v^\infty \quad \supset \quad \Pi_v^{(K_v)}$$

- Π_v: irreducible Hilbert space representation of $\mathrm{GL}_{n+1}(F_v)$
- Π_v^∞: smooth vectors, irreducible Fréchet representation of $\mathrm{GL}_{n+1}(F_v)$
- $\Pi_v^{(K_v)}$: K_v-finite vectors, irreducible $(\mathfrak{g}, K)$-module

Conceptually, consider the pair (Π, Σ) as an automorphic representation $\widetilde{\Pi}$ of

$$\widetilde{G} := \mathrm{GL}(n+1) \times \mathrm{GL}(n).$$

$\rightsquigarrow$ $\widetilde{\Pi}_v^\infty = \Pi_v^\infty \widehat{\otimes} \Sigma_v^\infty$ (completed projective tensor product)

$\rightsquigarrow$ $\widetilde{\Pi}_v^{(K_v \times K'_v)} = \Pi_v^{(K_v)} \otimes \Sigma_v^{(K'_v)}$ (algebraic tensor product)

Local L-functions: Archimedean case

Originally, Jacquet and Shalika define $L(s, \Pi_v \times \Sigma_v)$ via LLC.
Results of Jacquet-Shalika, Cogdell-Piatetski-Shapiro and Stade show:

- For $\Re(s) \gg 0$, $\Psi(s; -, -)$ extends to a continuous map

$$\mathscr{W}(\Pi_v^\infty, \psi_v) \widehat{\otimes} \mathscr{W}(\Sigma_v^\infty, \psi_v^{-1}) \ \to \ \mathbf{C}$$

- For every $\widetilde{W}_v \in \mathscr{W}(\Pi_v^\infty, \psi_v) \widehat{\otimes} \mathscr{W}(\Sigma_v^\infty, \psi_v^{-1})$ there is an *entire* function $H(s)$ with

$$\Psi(s; \widetilde{W}_v) = H(s) \cdot L(s, \Pi_v \times \Sigma_v)$$

- If $\widetilde{W}$ is K-finite, then $H(s)$ is a *polynomial* in s.
- For every $s_0 \in \mathbf{C}$ there is a K-finite $\widetilde{W}_v = W_v \otimes W'_v$ with $H(s_0) \neq 0$.
- Therefore, we find a K-finite $\widetilde{W}_v \in \mathscr{W}(\Pi_v, \psi_v)^{(K_v)} \otimes \mathscr{W}(\Sigma_v, \psi_v^{-1})^{(K'_v)}$ with

$$\Psi(s; \widetilde{W}_v) \ = \ L(s, \Pi_v \times \Sigma_v)$$

Caution: This is **not known** for $\mathrm{GL}(m) \times \mathrm{GL}(n)$!

Global Rankin-Selberg L-functions

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Put for $\Re(s) \gg 0$:

$$\Lambda(s, \Pi \times \Sigma) \;:=\; \prod_v L(s, \Pi_v \times \Sigma_v) \;=\; L(s, \Pi_\infty \times \Sigma_\infty) \cdot L(s, \Pi \times \Sigma)$$

Then:

- $\Lambda(s, \Pi \times \Sigma)$ extends *holomorphically* to all $s \in \mathbf{C}$, bounded in vertical strips
- $\Lambda(s, \Pi \times \Sigma)$ satisfies a functional equation

$$\Lambda(s, \Pi \times \Sigma) \;=\; \varepsilon(s, \Pi \times \Sigma)\Lambda(1 - s, \Pi^\vee \times \Sigma^\vee)$$

- There is are finitely many K-finite cusp forms $\varphi_i \in \Pi$, $\varphi_i' \in \Sigma$, with

$$\Lambda(s, \Pi \times \Sigma) \;=\; \sum_i I(s; \varphi_i, \varphi_i')$$

  and we may assume: $W_{\varphi_i} \;=\; \otimes_v W_{i,v}, \quad W_{\varphi_i'} \;=\; \otimes_v W_{i,v}',$ with:
  - For almost all $v \nmid \infty$, $W_{\varphi,i,v}$, $W_{\varphi,i,v}'$ are the normalized spherical vectors
  - For all $v \mid \infty$, $W_{\varphi,i,v}$, $W_{\varphi,i,v}'$ are K-finite

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Regular algebraic representations

Let for $v \mid \infty$:

- $\widetilde{K}_v := Z(F_v)^0 \cdot K_v$
- $\mathfrak{g}_v := \mathbf{C} \otimes_{\mathbf{R}} \mathrm{Lie}(\mathrm{GL}_{n+1}(F_v)) = (\mathbf{C} \otimes_{\mathbf{R}} F_v)^{n+1 \times n+1}$
- L_v: an irreducible rational representation of $\mathrm{res}_{F_v/\mathbf{R}} \mathrm{GL}(n+1)$

Theorem (Speh, Enright, Vogan-Zuckerman)

Let Π_v be an irreducible admissible generic representation of $\mathrm{GL}_{n+1}(F_v)$. TFAE:

- $H^\bullet(\mathfrak{g}_v, \widetilde{K}_v; \Pi_v \otimes L_v) \neq 0$
- $\Pi_v^{(K_v)} \cong A_{\mathfrak{q}}(\lambda)$ *is a cohomologically induced standard module with $\mathfrak{q} \subseteq \mathfrak{g}_v$ a ϑ-stable Borel subalgebra and $\lambda = H^0(\mathfrak{u}_{n+1}; L_v^\vee)$*
-

$$\Pi_v \cong \begin{cases} J_\lambda = \mathrm{Ind}_{P_{n+1}(\mathbf{R})}^{\mathrm{GL}_{n+1}(\mathbf{R})} \left[D(\ell_1) |\det|^{\mathbf{w}/2} \otimes D(\ell_2) |\det|^{\mathbf{w}/2} \otimes \cdots \right], \\ \ell = 2\lambda + 2\rho_{n+1} - \mathbf{w} : \ \mathbf{w} = \lambda_i + \lambda_{n+2-i}, & \text{if } F_v \cong \mathbf{R} \\ J_\lambda = \mathrm{Ind}_{B_{n+1}(\mathbf{R})}^{\mathrm{GL}_{n+1}(\mathbf{R})} \left[z^{a_1} \bar{z}^{b_1} \otimes \cdots \otimes z^{a_{n+1}} \bar{z}^{b_{n+1}} \right], \ \lambda = (\lambda^\iota, \lambda^{\bar{\iota}}), \\ a = \lambda^\iota + \rho_{n+1}, \ b = \lambda^{\bar{\iota}} - \rho_{n+1} = \mathbf{w} - \lambda^\iota - \rho_{n+1}, & \text{if } F_v \cong \mathbf{C} \end{cases}$$

Regular algebraic representations

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Definition (Clozel)

A cuspidal automorphic representation Π of $\mathrm{GL}_{n+1}(\mathbf{A}_F)$ is called *regular algebraic* if for all $v \mid \infty$ there is an L_v such that

$$H^\bullet(\mathfrak{g}_v, \widetilde{K}_v; \Pi_v \otimes L_v) \neq 0.$$

Define:

- $G_{n+1} := \mathrm{res}_{F/\mathbf{Q}} \mathrm{GL}(n+1)$
- S_{n+1} : maximal $\mathbf{Q}$-split torus in the center $Z_{n+1} \subseteq G_{n+1}$
- $K_{n+1} := \prod_{v|\infty} K_v$
- $\widetilde{K}_{n+1} := S_{n+1}(\mathbf{R})^0 \cdot K_{n+1} = \mathbf{R}_{>0} \cdot \prod_{v|\infty} K_v$
- For $K \subseteq G_{n+1}(\mathbf{A}^{(\infty)})$ compact open:

$$\mathscr{X}_{n+1}(K) \ := \ G_{n+1}(\mathbf{Q}) \backslash G_{n+1}(\mathbf{A}) / \widetilde{K}_{n+1} \cdot K$$

- L: rational representation of G_{n+1}
- $\underline{L}$: sheaf on $\mathscr{X}_{n+1}(K)$ attached to L

Automorphic cohomology

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Π is regular algebraic iff there is a rational representation L of G_{n+1} satisfying:

- There is $\mathbf{w} \in \mathbf{Z}$:
$$L \cong L^{c,\vee} \otimes (N_{F/\mathbf{Q}} \circ \det)^{\otimes \mathbf{w}}$$

- There is a $G_{n+1}(\mathbf{A}^{(\infty)})$-module embedding
$$\Pi^{(\infty)} \subseteq \varinjlim_K H^\bullet(\mathscr{X}_{n+1}(K); \underline{L})$$

If this is the case, we have in fact

$$H^\bullet(\mathfrak{g}_{n+1}, \widetilde{K}_{n+1}; \Pi \otimes L) \subseteq \varinjlim_K H_!^\bullet(\mathscr{X}_{n+1}(K); \underline{L})$$

where *inner cohomology* is defined as

$$H_!^\bullet(\mathscr{X}_{n+1}(K); \underline{L}) := \mathrm{image}\left[H_c^\bullet(\mathscr{X}_{n+1}(K); \underline{L}) \to H^\bullet(\mathscr{X}_{n+1}(K); \underline{L})\right]$$

Automorphic cohomology

We observe:

$$
\begin{aligned}
H^\bullet(\mathfrak{g}_{n+1}, \widetilde{K}_{n+1}; \Pi \otimes L) &= H^\bullet(\mathfrak{g}_{n+1}, \widetilde{K}_{n+1}; \Pi_\infty \otimes L) \otimes \Pi^{(\infty)} \\
&= \left[\bigwedge^{\bullet} (\mathfrak{g}_{n+1} / \widetilde{\mathfrak{k}}_{n+1})^* \otimes \Pi_\infty \otimes L \right]^{\widetilde{K}_{n+1}} \otimes \Pi^{(\infty)} \\
&= \bigwedge^{\bullet - \dim(\mathfrak{n} \cap \mathfrak{p})} (\mathfrak{l} \cap \widetilde{\mathfrak{k}}_{n+1})^* \otimes \Pi^{(\infty)}
\end{aligned}
$$

for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n} \subseteq \mathfrak{g}_{n+1} = \mathfrak{p} \oplus \widetilde{\mathfrak{k}}_{n+1}$ a ϑ-stable Borel subalgebra.
In particular: The multiplicity of $\Pi^{(\infty)}$ in *bottom degree*

$$
b_{n+1} := \dim(\mathfrak{n} \cap \mathfrak{p}) = r_1 \cdot \left\lfloor \frac{(n+1)^2}{4} \right\rfloor + r_2 \cdot \frac{(n+1)n}{2}
$$

is **one**. Furthermore, there's the numerical **miracle** (S., KMS, J.)

$$
b_{n+1} + b_n = \dim \mathscr{Y}_n(K'), \quad \mathscr{Y}_n(K') := G_n(\mathbf{Q}) \backslash G_n(\mathbf{A}) / K_n \cdot K'
$$

Automorphic cohomology classes

Departing from $W^{(\infty)} \in \mathscr{W}(\Pi^{(\infty)}, \psi^{(\infty)})^K$ we obtain a de Rham cocycle

$$\sum_i \omega_i \otimes W_{\infty,i} \otimes W^{(\infty)} \otimes l_i \in \left[\bigwedge^{b_{n+1}} (\mathfrak{g}_{n+1}/\widetilde{\mathfrak{k}}_{n+1})^* \otimes \mathscr{W}(\Pi, \psi)^K \otimes L \right]^{\widetilde{K}_{n+1}}$$

By Fourier inversion, writing $W_{\varphi_i} = W_{\infty,i} \otimes W^{(\infty)}$, get a cohomology class

$$\begin{aligned} \sum_i \omega_i \otimes \varphi_i \otimes l_i &\in \left[\bigwedge^{b_{n+1}} (\mathfrak{g}_{n+1}/\widetilde{\mathfrak{k}}_{n+1})^* \otimes \Pi^K \otimes L \right]^{\widetilde{K}_{n+1}} \\ &\subseteq H_c^{b_{n+1}}(\mathscr{X}_{n+1}(K); \underline{L}) \end{aligned}$$

Likewise, any $W'^{(\infty)} \in \mathscr{W}(\Sigma^{(\infty)}, \psi^{(\infty),-1})^{K'}$ yields a cuspdial cohomology class

$$\sum_j \omega'_j \otimes \varphi'_j \otimes l'_j \in H_c^{b_n}(\mathscr{X}_n(K'); \underline{L}')$$

Evaluation of cohomology classes

Put $G := G_{n+1} \times G_n$.
Observe that for $L \subseteq G_n(\mathbf{A}^{(\infty)})$ compact open and sufficiently small:

$$\begin{array}{ccc} \mathscr{Y}_n(L) & \longrightarrow & \mathscr{X}_{n+1}(K) \times \mathscr{X}_n(K') \\ \| & & \| \\ \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F)/K_n \cdot L & \xrightarrow[g \mapsto (\mathrm{diag}(g,1),g)]{} & G(\mathbf{Q})\backslash G(\mathbf{A})/(\widetilde{K}_{n+1}K \times \widetilde{K}_n K') \end{array}$$

is *proper*. Together with Poincaré duality we obtain a rationality preserving map

$$\begin{aligned} H_c^{b_{n+1}}(\mathscr{X}_{n+1}(K);\underline{L}) \otimes H_c^{b_n}(\mathscr{X}_n(K');\underline{L}') &\to H_c^{\dim \mathscr{Y}_n}(\mathscr{Y}_n(L);\underline{L}\otimes\underline{L}') \\ &\cong H^0(\mathscr{Y}_n(L);\underline{L}^\vee \otimes \underline{L}'^\vee) \\ &= (L^\vee \otimes L'^\vee)^{\Gamma_L} \qquad [\Gamma_L = \mathrm{GL}_n(F) \cap L] \\ &\supseteq \sum_{\nu \in \mathbf{Z}} \underbrace{\mathrm{Hom}_{G_n}(L \otimes L', (N_{F/\mathbf{Q}} \circ \det)^{\otimes \nu})}_{\dim \leq 1} \end{aligned}$$

Evaluation of cohomology classes

For the constructed automorphic cohomology classes this yields

$$\sum_{i,j}(\omega_i\otimes\varphi_i\otimes l_i)\otimes(\omega_j'\otimes\varphi_j'\otimes l_j') \;\mapsto$$

$$\sum_{i,j}\langle\omega^{G_n},(\omega_i\wedge\omega_j')|_{\mathscr{Y}_n}\rangle\cdot\Big(\eta_\nu(l_i\otimes l_j')\cdot I(\tfrac{1}{2}+\nu;\varphi_i,\varphi_j')\Big)_\nu$$

Here:

- $\omega^{G_n}\in\bigwedge^{\dim\mathscr{Y}_n}\mathfrak{g}_n/\mathfrak{k}_n$: a generator (an invariant measure on $G_n(\mathbf{R})/K_n$)
- $\langle -,-\rangle:\bigwedge^{\dim\mathscr{Y}_n}\mathfrak{g}_n/\mathfrak{k}_n\otimes\bigwedge^{\dim\mathscr{Y}_n}(\mathfrak{g}_n/\mathfrak{k}_n)^*\to\mathbf{C}$ the canonical pairing
- ν runs through the integers satisfying

$$\mathrm{Hom}_{G_n}(L\otimes L',(N_{F/\mathbf{Q}}\circ\det)^{\otimes\nu})\neq 0$$

If existent, those are in bijection with the critical values of $L(s,\Pi\times\Sigma)$ (Kasten-Schmidt, Raghuram)
Recall: $s_0\in\frac{1}{2}+\mathbf{Z}$ *critical* for $L(s,\Pi\times\Sigma)$ $\Leftrightarrow$
neither $L(s,\Pi_\infty\times\Sigma_\infty)$ nor $L(1-s,\Pi_\infty^\vee\times\Sigma_\infty^\vee)$ have a pole at $s=s_0$

Evaluation of cohomology classes

For each ν the ζ-integral $I(\frac{1}{2}+\nu;\varphi_i,\varphi'_j)$ evaluates to

$$\int\limits_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbf{A}_F)} \varphi_i\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\right)\varphi'_j(g)|\det g|^{\nu}dg \quad = \quad L(\tfrac{1}{2}+\nu,\Pi\times\Sigma)$$

$$\cdot\prod_{v\nmid\infty}\left[\frac{\Psi(s;W_v,W'_v)}{L(s,\Pi_v\times\Sigma_v)}\right]_{s=\frac{1}{2}+\nu}\cdot\underbrace{\prod_{v\mid\infty}\Psi(\tfrac{1}{2}+\nu;W_{v,i},W'_{v,j})}_{\text{contributes to archimedean period}}$$

- For almost all $v\nmid\infty$ and all $s_0\in\mathbf{C}$:

$$\left[\frac{\Psi(s;W_v,W'_v)}{L(s,\Pi_v\times\Sigma_v)}\right]_{s=s_0} = 1$$

- For the other $v\nmid\infty$ we may hope to choose W_v, W'_v ‘suitably’
- **Problem:** For $v\mid\infty$ the test vectors $W_{v,i}$ and $W'_{v,j}$ are **not** known explicitly.
  (For $n\geq 2$ there is no explicit realization of Π_v and Σ_v known)

Non-vanishing of archimedean periods

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Theorem (Kasten-Schmidt, Int. J. Number Theory 2013)
If $n = 2$ and $F/\mathbf{Q}$ totally real, then for all $s_0 \in \mathbf{C}$, $\delta \in \{0,1\}^{[F:\mathbf{Q}]}$:

$$\dim \mathrm{Hom}_{\mathfrak{g}_n, K_n}\left(\Pi_\infty^{(K_{n+1})} \otimes \Sigma_\infty^{(K_n)}, \mathrm{sgn}_\infty^\delta \otimes |\cdot|^{-s_0}\right) \leq 1$$

Theorem (Aizenbud-Gourevich-Sayag, Comp. Math. 2008; Sun-Zhu, Ann. Math. 2012)
For all $n \geq 1$, $F/\mathbf{Q}$ arbitrary, and all $s_0 \in \mathbf{C}$, $\delta \in \{0,1\}^{r_1}$:

$$\dim \mathrm{Hom}_{G_n(\mathbf{R})}\left(\Pi_\infty^\infty \widehat{\otimes} \Sigma_\infty^\infty, \mathrm{sgn}_\infty^\delta \otimes |\cdot|^{-s_0}\right) \leq 1$$

Theorem (B. Sun, JAMS 2017)
For all $n \geq 1$, $F/\mathbf{Q}$ arbitrary and all $s_0 = \frac{1}{2} + \nu$ critical for $L(s, \Pi \times \Sigma)$:

$$\sum_{i,j} \langle \omega^{G_n}, (\omega_i \wedge \omega_j')|_{\mathscr{Y}_n} \rangle \cdot \eta_\nu(I_i \otimes I_j') \cdot \prod_{v|\infty} \Psi(\tfrac{1}{2}+\nu; W_{v,i}, W_{v,j}') \neq 0$$

Rationality of special values

Theorem (Schmidt, KMS, Kasten-Schmidt, Raghuram-Shahidi, J., Raghuram, Sun)

Let $F/\mathbf{Q}$ be arbitrary, Π and Σ regular algebraic of balanced *cohomological weight. Then there are periods $\Omega_\nu^\varepsilon \in \mathbf{C}^\times$ with the following property: For each finite order Hecke character $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$, and each $s = \frac{1}{2} + \nu$ critical for $L(s, \Pi \times \Sigma)$:*

$$\frac{L(\frac{1}{2}+\nu, \Pi \times (\Sigma \otimes \chi))}{G(\chi)^{\frac{(n+1)n}{2}} \Omega_\nu^{(-1)^\nu \operatorname{sgn}\chi}} \in \mathbf{Q}(\Pi, \Sigma, \chi).$$

Theorem (Hida, Duke 1994; J., TAMS 2018)

Let $1 \leq n \leq 2$ and F totally real if $n = 2$ or $F/\mathbf{Q}$ and n arbitrary and assume the continuity of cohomologically induced functionals. Then for all $s_0 = \frac{1}{2} + \nu_0$ and $s_1 = \frac{1}{2} + \nu_1$ critical for $L(s, \Pi \times \Sigma)$ and all signs ε:

$$\frac{\Omega_{\nu_0}^{(-1)^{\nu_0}\varepsilon}}{\Omega_{\nu_1}^{(-1)^{\nu_1}\varepsilon}} \in \frac{(2\pi i)^{[F:\mathbf{Q}]\nu_0}}{(2\pi i)^{[F:\mathbf{Q}]\nu_1}} \cdot \mathbf{Q}(\Pi, \Sigma).$$

Remark: If $F/\mathbf{Q}$ is an imaginary CM field Grobner-Lin obtained a similar relation under different hypotheses.

Rationality of special values

Theorem (J., Math. Ann. 2017, TAMS 2018)

Let $F/\mathbf{Q}$ be arbitrary, Π and Σ regular algebraic of balanced *cohomological weight. Then:*

- *The* $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbf{A}^{(\infty)})$*-module* $\Pi^{(K_\infty)} \otimes \Sigma^{(K'_\infty)}$ *admits a* $\mathbf{Q}(\Pi, \Sigma)$*-rational structure, unique up to scalars*
- *As* $(\mathfrak{g}_\infty, K^0_\infty) \times G(\mathbf{A}^{(\infty)})$*-module:* $\Pi^{(K_\infty)} \otimes \Sigma^{(K'_\infty)} = \bigoplus_\varepsilon \widetilde{\Pi}^\varepsilon_\infty$ *and each* $\widetilde{\Pi}^\varepsilon_\infty$ *admits a* $\mathbf{Q}(\Pi, \Sigma, i^{\frac{(n+1)n}{2}})$*-rational structure, unique up to scalars*

Put $E := \mathbf{Q}(\Pi, \Sigma, i^{\frac{(n+1)n}{2}})$ and define two E-rational structures:

- $\widetilde{\Pi}^{\mathrm{B}}_\infty$: via $H^b(\mathscr{X}(K); \underline{L}_E \otimes \underline{L}'_E)$
- $\widetilde{\Pi}^{\mathrm{dR}}_\infty$: via $H^b(\mathfrak{g}_\infty; \widetilde{K}_\infty; \widetilde{\Pi}^+_E \otimes L_E \otimes L'_E)$ using E-rational test vectors

We have the analogy:

$$H^b(\mathscr{X}(K); \underline{L}_E \otimes \underline{L}'_E)[\widetilde{\Pi}^\varepsilon] \otimes_E \mathbf{C} \xrightarrow{\sim} H^b(\mathfrak{g}_\infty; \widetilde{K}_\infty; \widetilde{\Pi}^{+,K}_E \otimes L_E \otimes L'_E) \otimes_E \mathbf{C}$$

$$H_{\mathrm{B}}(M(\Pi) \otimes M(\Sigma))^+ \otimes_E \mathbf{C} \xrightarrow{\sim} H_{\mathrm{dR}}(M(\Pi) \otimes M(\Sigma)) / F^0 \otimes_E \mathbf{C}$$

Towards p-adic L-functions

Naively, given an L-series $\sum_{n=1}^{\infty} a_n n^{-s}$, one may chop it into pieces

$$\mu_\nu(x + (p^\beta)) := \sum_{n \equiv x\,(p^\beta)} \frac{a_n}{n^s}$$

and obtain for given s a $\mathbf{C}$-valued distribution μ. Integration against characters yields

$$\int \chi(x) d\mu_\nu(x) = \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n^\nu}$$

Automorphically this means rewriting:

$$I(s, \varphi_i, \varphi_i') = \sum_{x \in C(p^\beta)} \int_{\det^{-1}(x)} \varphi_i(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) \varphi_j'(g) |\det g|^\nu dg$$

Therefore, one may define

$$\mu(x + (p^\beta)) := \int_{\det^{-1}(x)} \varphi_i(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) \varphi_j'(g) |\det g|^\nu dg$$

Towards p-adic L-functions

The cohomological interpretation of this intuition is:

$$\begin{array}{ccc} \mathscr{Y}_n(L_\beta) & = & \bigsqcup_{x\in C(p^\beta)} \overbrace{\Gamma_x\backslash \mathrm{GL}_n(F\otimes_{\mathbf{Q}}\mathbf{R})/K_n}^{=\det^{-1}(x)} \\ {\scriptstyle \det}\downarrow & & \downarrow \\ C(p^\beta) & = & C(p^\beta) \end{array}$$

where $L_\beta \subseteq \mathrm{GL}_n(\mathbf{A}_F^{(\infty)})$ is a suitable compact open such that

$$C(p^\beta) \;=\; F^\times\backslash \mathbf{A}_F^\times / F_{\mathbf{R}}^+ \det(L_\beta)$$

Then we may define a distribution by integrating cohomology classes separately over the fibers of the determinant.

Table of Contents

Some history

- **1993:** Claus Schmidt constructs p-adic **measures** for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ over **Q** for trivial cohomological weights, spherical and ordinary at p, **non-vanishing** of archimedean period due to Barry Mazur
- **2000:** Kazhdan-Mazur-Schmidt construct a p-adic **distribution** for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ over **Q** for representations spherical and ordinary at p, **not** bounded, archimedean non-vanishing hypothesis explicitly formulated
- **2001:** Schmidt improves the KMS construction to obtain a p-adic **measure** in the ordinary case but has to exclude $p <= n+1$
- **2005:** Utz extends Schmidt's construction to all p for $n = 2, 3, 4$
- **2009:** **all** ordinary p for all n, arbitrary number fields (J.)
- **2013:** Kasten-Schmidt prove **non-vanishing** of periods for $\mathrm{GL}(3) \times \mathrm{GL}(2)/\mathbf{Q}$
- **2014:** arbitrary coh. weights over totally real fields, **functional equation** (J.)
- **2015:** arbitrary coh. weights over arbitrary number fields (J.)
- **2017:** Sun proves **non-vanishing** of archimedean periods in **all** cases
- **2017:** allow for near ordinarity at p, **Manin congruences**, extend to Hida's universal nearly ordinary cohomology for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ (J.)

To be continued.

Thank you for your attention.