Cohomological Hall algebras for quiver with potential

Sven Meinhardt (jointly with Ben Davison)

Oaxaca

November 3, 2018

Moduli of quiver representations

Quiver $Q=(Q_0,Q_1,h,t)$, dimension vector $d=(d_v)_{v\in Q_0}\in\mathbb{N}^{Q_0}$

$$\mathrm{Rep}_d(Q):=\prod_{Q_1\ni\alpha:v\to w}\mathrm{Mat}_{d_w\times d_v}(\mathbb{C})\;\cong\;\prod_{Q_1\ni\alpha:v\to w}\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_v},\mathbb{C}^{d_w})$$

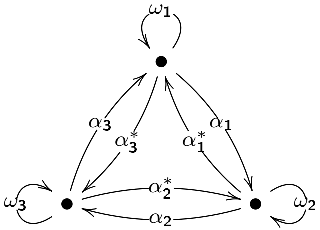
with action of $G_d:=\prod_{v\in Q_0}\mathrm{GL}_{d_v}(\mathbb{C})$ by simultaneous conjugation.

Potential $W=$ linear combination of cycles in Q (up to cyclic order)

$\rightsquigarrow \partial W/\partial\alpha=0$ relations for representations

Example

Given Q, consider the quiver Q^{ex} with $Q_0^{ex} = Q_0$ and $Q_1^{ex} = Q_1 \sqcup Q_1^* \sqcup Q_0$



with potential $W^{ex} = \sum_{v \in Q_0} \omega_v \Big(\sum_{\alpha: w \to v} \alpha\alpha^* - \sum_{\alpha: v \to w} \alpha^*\alpha \Big)$.
Hence

$$\begin{aligned}
\partial W^{ex}/\partial \omega_v &= \sum_{\alpha: w \to v} \alpha\alpha^* - \sum_{\alpha: v \to w} \alpha^*\alpha, \\
\partial W^{ex}/\partial \alpha^* &= \omega_{h(\alpha)}\alpha - \alpha\omega_{t(\alpha)}, \\
\partial W^{ex}/\partial \alpha &= \alpha^*\omega_{t(\alpha^*)} - \omega_{h(\alpha^*)}\alpha^*.
\end{aligned}$$

Question: Why are relations induced by potentials good?

1. There is a G_d-invariant function $\mathrm{Tr}_d(W) : \mathrm{Rep}_d(Q) \longrightarrow \mathbb{C}$ such that $\mathrm{Crit}(\mathrm{Tr}_d(W))$ is the space of representations satisfying the relations $\partial W/\partial\alpha = 0$ for all $\alpha \in Q_1$.
2. There exists a perverse sheaf of vanishing cycles $\phi_{\mathrm{Tr}_d(W)}$ on $\mathrm{Crit}(\mathrm{Tr}_d(W))$ measuring the singularities of the fibers of $\mathrm{Tr}_d(W)$.

Stability conditions:
For $(\xi_v)_{v\in Q_0} \in \mathbb{H}_+^{Q_0}$ we get a stability condition σ with central charge $Z(M) = \sum_{v\in Q_0} \xi_v \cdot \dim M_v$ and standard t-structure.
Let $\mathrm{Rep}_d^{st} \subset \mathrm{Rep}_d^{ss}(Q) \subset \mathrm{Rep}_d(Q)$ be the open subsets of σ-stable and σ-semistable representations.

Absolute cohomological Hall algebra

Fix a "phase" $\vartheta \in (0, \pi)$ and introduce the shorthand
$\Gamma_\vartheta := \{0 \neq d \in \mathbb{N}^{Q_0} \mid \arg(\sum_{v \in Q_0} \xi_v \cdot d_v) = \vartheta\} \cup \{0\}$

Definition

We define the **absolute cohomological Hall algebra**

$$\boldsymbol{HA}^*_\vartheta(Q, W, \sigma) := \bigoplus_{d \in \Gamma_\vartheta} \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i_{G_d}\left(\mathrm{Rep}^{ss}_d(Q), \phi_{\mathrm{Tr}_d(W)}\right).$$

by taking the G_d-equivariant cohomology. The product is induced by a correspondence diagram.

Notice:
The absolute Hall algebra $\boldsymbol{HA}^*_\vartheta(Q, W, \sigma) = \bigoplus_{d \in \Gamma_\vartheta} \boldsymbol{HA}^*_d(Q, W, \sigma)$ is a bi-graded vector space.

The product

Given dimension vectors $d', d'' \in \Gamma_\vartheta$, consider

$$\mathrm{Rep}^{ss}_{d',d''}(Q) := \left\{(M_\alpha) \in \mathrm{Rep}^{ss}_{d'+d''}(Q) \mid M_\alpha \text{ upper block triagonal}\right\}$$

with its action by the subgroup $G_{d',d''} \subset G_{d'+d''}$ of upper block triagonal invertible matrices.
Get equivariant maps

$$\begin{tikzcd}
& \mathrm{Rep}^{ss}_{d',d''}(Q) \arrow[dl, "\pi_1 \times \pi_3"'] \arrow[dr, hook, "\pi_2"] & \\
\mathrm{Rep}^{ss}_{d'}(Q) \times \mathrm{Rep}^{ss}_{d''}(Q) & & \mathrm{Rep}^{ss}_{d'+d''}(Q)
\end{tikzcd}$$

The Hall algebra product is essentially $\pm(\pi_2)_* \circ (\pi_1 \times \pi_3)^*$

$$\boldsymbol{HA}^*_{d'}(Q, W, \sigma) \otimes \boldsymbol{HA}^*_{d''}(Q, W, \sigma) \longrightarrow \boldsymbol{HA}^*_{d'+d''}(Q, W, \sigma).$$

Quantum groups

Theorem (Kontsevich–Soibelman, Davison–M.)

1. *The absolute cohomological Hall algebra* $\boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$ *is associative with unit.*
2. *The absolute cohomological Hall algebra* $\boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$ *has a compatible (localized) coproduct turning* $\boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$ *into a (localized) bi-algebra.*
3. *The absolute cohomological Hall algebra* $\boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$ *has a compatible increasing filtration turning* $\boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$ *into a filtered algebra.*

Question: What can we say about the structure of the absolute cohomological Hall algebra or its associated graded $\mathfrak{gr}_* \boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$?

Moduli spaces

Theorem (King)

For all σ and all d, the subset $\mathrm{Rep}_d^{ss}(Q) \subset \mathrm{Rep}_d(Q)$ is the open subvariety of semistable points for a suitable linearization χ of the G_d-action on $\mathrm{Rep}_d^{ss}(Q)$. Moreover

$$\mathcal{M}_d^{ss}(Q) := \mathrm{Rep}_d^{ss}(Q) /\!\!/_{\chi} G_d$$

is a quasiprojective variety parameterizing σ-semistable representations up to S-equivalence.

Here, $M \sim_S M'$ if M and M' have the same stable subquotients (up to isomorphism) counted with multiplicities.
$\mathcal{M}_d^{st}(Q) := \mathrm{Rep}_d^{st}(Q) /\!\!/_{\chi} G_d$ is either empty or dense in $\mathcal{M}_d^{ss}(Q)$.

Relative cohomological Hall algebra

Denote by $\mathfrak{q} : \mathrm{Rep}_d^{ss}(Q) \twoheadrightarrow \mathcal{M}_d^{ss}(Q)$ the quotient map.

Definition

We define the **relative cohomological Hall algebra** by

$$\boldsymbol{\mathcal{HA}}_\vartheta^*(Q, W, \sigma) := \bigoplus_{d \in \Gamma_\vartheta} \bigoplus_{i \in \mathbb{Z}} \mathrm{R}^i \mathfrak{q}_{G_d}\left(\phi_{\mathrm{Tr}_d(W)}\right),$$

where $\mathrm{R}^i\mathfrak{q}_G$ is the i-th direct G_d-equivariant image with respect to the perverse t-structure on $\mathcal{M}_d^{ss}(Q)$.

Notice:
The relative cohomological Hall algebra $\boldsymbol{\mathcal{HA}}_\vartheta^*(Q, W, \sigma) = \bigoplus_{d \in \Gamma_\vartheta} \boldsymbol{\mathcal{HA}}_d^*(Q, W, \sigma)$ is a bi-graded perverse sheaf on $\mathcal{M}_\vartheta^{ss}(Q) := \sqcup_{d \in \Gamma_\vartheta} \mathcal{M}_d^{ss}(Q)$.

Recall:

1. $\boldsymbol{HA}^*_d(Q,W,\sigma) = \mathrm{H}^*_{G_d}\left(\mathrm{Rep}^{ss}_d(Q), \phi_{\mathrm{Tr}_d(W)}\right)$ is a graded vector space.
2. $\boldsymbol{\mathcal{HA}}^*_d(Q,W,\sigma) = \mathrm{R}^*\mathfrak{q}_{G_d}\left(\phi_{\mathrm{Tr}_d(W)}\right)$ graded perverse sheaf on $\mathcal{M}^{ss}_d(Q)$.

There is a "perverse" filtration on $\boldsymbol{HA}^*_d(Q,W,\sigma)$ and a "perverse" Leray spectral sequence with E_2-term

$$\mathrm{H}^i\left(\mathcal{M}^{ss}_d(Q), \boldsymbol{\mathcal{HA}}^j_d(Q,W,\sigma)\right)$$

converging to $\mathfrak{gr}_j\, \boldsymbol{HA}^{i+j}_d(Q,W,\sigma)$.

Proposition (Davison–M.)

The spectral sequence collapses at E_2, i.e. $\forall i,j \in \mathbb{Z}$

$$\mathfrak{gr}_j\, \boldsymbol{HA}^{i+j}_d(Q,W,\sigma) \cong \mathrm{H}^i\left(\mathcal{M}^{ss}_d(Q), \boldsymbol{\mathcal{HA}}^j_d(Q,W,\sigma)\right).$$

We can extend the previous diagram

$$\begin{tikzcd}
 & \mathfrak{Rep}^{ss}_{d',d''}(Q) \arrow[dl, "\pi_1\times\pi_3"'] \arrow[dr, "\pi_2"] & \\
\mathfrak{Rep}^{ss}_{d'}(Q)\times\mathfrak{Rep}^{ss}_{d''}(Q) \arrow[d, "\mathtt{q}\times\mathtt{q}"'] & & \mathfrak{Rep}^{ss}_{d'+d''}(Q) \arrow[d, "\mathtt{q}"] \\
\mathcal{M}^{ss}_{d'}(Q)\times\mathcal{M}^{ss}_{d''}(Q) \arrow[rr, "\oplus"] & & \mathcal{M}^{ss}_{d'+d''}(Q)
\end{tikzcd}$$

Using adjunction morphisms for pull-back and push-forwards, the Thom–Sebastiani isomorphism and properties of the vanishing cycle functor, we get maps
$\oplus_*\Big(\boldsymbol{\mathcal{HA}}_{d'}(Q,W,\sigma)\boxtimes\boldsymbol{\mathcal{HA}}_{d''}(Q,W,\sigma)\Big)\longrightarrow\boldsymbol{\mathcal{HA}}_{d'+d''}(Q,W,\sigma)$ of perverse sheaves.

Summing over $d', d'' \in \Gamma_\vartheta$ we get an algebra in an appropriate symmetric monoidal tensor category.

Theorem (Davison–M.)

1. *The relative cohomological Hall algebra $\mathcal{HA}^*_\vartheta(Q, W, \sigma)$ is associative with unit and induces the same structure on its (hyper)cohomology.*
2. *The collapsing spectral sequence is a spectral sequence of algebras inducing an isomorphism of algebras*

$$\mathfrak{gr}_* \, \boldsymbol{HA}^*_\vartheta(Q, W, \sigma) \cong \mathrm{H}^* \left(\mathcal{M}^{ss}_\vartheta(Q), \mathcal{HA}^*_\vartheta(Q, W, \sigma)\right).$$

Question: Why is this useful?

Definition

We call a stability condition σ **symmetric** if for all $\vartheta \in (0,\pi)$ and all $d', d'' \in \Gamma_\vartheta$ the bilinear pairing $\sum_{\alpha:v\to w} d'_v d''_w$ is symmetric.

Remark: Non-symmetric stability conditions are contained in the union of countably many walls. Hence, a **generic** stability condition is symmetric.

Theorem (Davison–M.)

For a symmetric stability condition σ and any phase $\vartheta \in (0,\pi)$ the relative Hall algebra $\boldsymbol{\mathcal{HA}}^*_\vartheta(Q,W,\sigma)$ is a symmetric algebra, i.e.

$$\boldsymbol{\mathcal{HA}}^*_\vartheta(Q,W,\sigma) = \mathrm{Sym}(\mathcal{G}^*_\vartheta)$$

for some bi-graded perverse sheaf $\mathcal{G}^*_\vartheta$ on $\mathcal{M}^{ss}_\vartheta(Q) = \sqcup_{d\in\Gamma_\vartheta} \mathcal{M}^{ss}_d(Q)$.

Remark: The absolute Hall algebra $\boldsymbol{HA}^*_\vartheta(Q,W,\sigma)$ is in general not (graded) commutative even for symmetric σ. But:

Corollary

For symmetric σ and any ϑ we conclude

$$\mathfrak{gr}_* \, \boldsymbol{HA}^*_\vartheta(Q,W,\sigma) \cong \mathrm{Sym}\left(\mathrm{H}^*\left(\mathcal{M}^{ss}_\vartheta(Q), \mathcal{G}^*_\vartheta\right)\right).$$

Question: Can we determine $\mathcal{G}^*_\vartheta$?

Notice: $\mathrm{Tr}_d(W) : \mathrm{Rep}_d^{ss}(Q) \xrightarrow{q} \mathcal{M}_d^{ss}(Q) \xrightarrow{f_d} \mathbb{C}$ for some function f_d.

Definition

1. For $d \in \mathbb{N}^{Q_0}$ we form the **BPS sheaf**

$$\mathcal{DT}_d(Q,W,\sigma) = \begin{cases} \phi_{f_d}(\mathcal{IC}_{\mathcal{M}_d^{ss}(Q)}(\mathbb{Q})) & \text{if } \mathcal{M}_d^{st}(Q) \neq \emptyset, \\ 0 & \text{else} \end{cases}$$

   $\mathcal{IC}_{\mathcal{M}_d^{ss}(Q)}(\mathbb{Q})$ is the intersection complex of $\mathcal{M}_d^{ss}(Q)$.
2. $\mathcal{DT}_\vartheta(Q,W,\sigma) := \bigoplus_{d\in\Gamma_\vartheta} \mathcal{DT}_d(Q,W,\sigma)$ a graded perverse sheaf on $\mathcal{M}_\vartheta^{ss}(Q,W)$.
3. $\mathrm{H}^*\left(\mathcal{M}_\vartheta^{ss}(Q), \mathcal{DT}_\vartheta(Q,W,\sigma)\right)$ is the **space of BPS states**. Its (refined) dimension is the (refined) **BPS invariant**.

Theorem (M.-Reineke, Davison–M.)

For a symmetric stability condition σ and any $\vartheta \in (0, \pi)$ we get

$$\mathcal{G}^*_\vartheta = \mathcal{DT}_\vartheta(Q, W, \sigma) \otimes \mathrm{H}^*(B\mathbb{C}^\times)_{vir} = \bigoplus_{i \in \mathbb{N}} \mathcal{DT}_\vartheta(Q, W, \sigma)[-2i-1].$$

Corollary

For symmetric σ and any ϑ the associated graded algebra $\mathfrak{gr}_* \boldsymbol{HA}^*_\vartheta(Q, W, \sigma)$ wrt. the perverse filtration is a symmetric algebra generated by $\mathrm{H}^*\left(\mathcal{M}^{ss}_\vartheta(Q), \mathcal{DT}_\vartheta(Q, W, \sigma)\right) \otimes \mathrm{H}^*(B\mathbb{C}^\times)_{vir}$.

Corollary

The commutator in $\boldsymbol{HA}^*_\vartheta(Q, W, \sigma)$ induces a graded Lie algebra structure on

$$\mathfrak{gr}_1 \boldsymbol{HA}^*_\vartheta(Q, W, \sigma) \cong \mathrm{H}^{*-1}\left(\mathcal{M}^{ss}_\vartheta(Q), \mathcal{DT}_\vartheta(Q, W, \sigma)\right).$$

Topology of moduli spaces

For $W = 0$, we get

$$\mathcal{DT}_d(Q, W, \sigma) = \begin{cases} \mathcal{IC}_{\mathcal{M}_d^{ss}(Q)}(\mathbb{Q}) & \text{if } \mathcal{M}_d^{st}(Q) \neq \emptyset, \\ 0 & \text{else.} \end{cases}$$

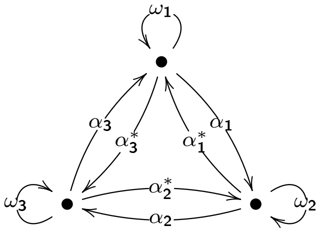
Thus, the BPS invariants compute intersection Euler characteristics and the refined BPS invariants the Poincaré/Hodge polynomials of the (compactly supported) intersection cohomology.

Example (Reineke): Consider the Jordan quiver $Q^{(g)}$ with g loops $\alpha_1,\dots,\alpha_g \circlearrowleft \bullet$. Then

$$\sum_i \dim \mathrm{IC}_c^i(\mathcal{M}_d(Q^{(g)}), \mathbb{C}) t^i = t^{(g-1)d^2+1} \frac{1 - t^{-2}}{1 - t^{-2d}} \sum_{C \in U_d^{ap}/C_d} t^{-2 \deg C}.$$

Kac–Moody algebras

Given Q $\rightsquigarrow$ Q^{ex}



with potential $W^{ex} = \sum_{v \in Q_0} \omega_v \left(\sum_{\alpha : w \to v} \alpha\alpha^* - \sum_{\alpha : v \to w} \alpha^*\alpha \right)$.

Theorem (BBS,Mozgovoy,HLR,Davison)

The refined BPS invariant is given by the Kac polynomial for Q and has positive coefficients (Kac conjecture).

Thank you!