Helly and Tverberg Type Theorems; Mass Partitions and Rado's Central Type Theorems in Geometry, Combinatorics and Topology IV.

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1 Introduction

Let's begin by reminding three classical results, the theorems of Carathéodory, Helly, and Radon's lemma, more than a hundred years old that have been at the heart of combinatorial convexity:
Helly's theorem on one hand; asserts that if $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$ in which every $d+1$ members have nonempty intersection, then some point lies in every set in $\mathcal{F}$. **Carathéodory's theorem** on the other, asserts that every point in the convex hull of a point set $P \subset \mathbb{R}^d$ is in the convex hull of a subset $Q \subset P$ with at most $d+1$ points. Finally, **Radon Lemma** ensures than any $d+2$ points in $\mathbb{R}^d$ can be partitioned into two classes with intersecting convex hulls.

Helly's theorem has stimulated numerous generalization and variants. In the past twenty years, there has been a significant increase in research activity and productivity in the area which was the aim of this workshop. During this week this workshop brought together senior and junior researchers in the area with the objective of interchanging ideas and assessing recent advances.

Another basic result is **Tverberg's theorem** (the r-partite version of Radon's Lemma) is more than fifty years old and is equally significant. This theorem states that every $(d+1)(r-1)+1$ points in Euclidean d-space $\mathbb{R}^d$ can be partitioned into r parts such that the convex hulls of these parts have nonempty intersection. As we will see next this theorem still remains central and is one of the most intriguing results of combinatorial geometry.

The workshop combined and interesting mixture of talks one problem session and many time for discussions in groups. During the week the academic interest was mainly centerer (among several other important things) on three important generalizations. The first one is the the Colorful Helly Theorem discovered by Lovász and reported by Bárány.
Colorful Helly. Let $F_1, \ldots, F_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. Suppose for every choice $K_1 \in F_1, \ldots, K_{d+1} \in F_{d+1}$ we have $\cap_1^{d+1} K_i \neq \emptyset$. Then for one of the families F_i we have $\cap_{K \in F_i} K \neq \emptyset$.

The second generalization of Helly's theorem is the Fractional Helly Theorem due to Katchalski and Liu.
Fractional Helly For every $d \geq 1$ and $\alpha \in (0,1)$ there exists a $\beta = \beta(\alpha, d) \in (0,1)$ with the following property: Let F be a family of $n > d+1$ convex sets in $\mathbb{R}^d$ and suppose at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples in F have nonempty intersection. Then there exists some βn members of F whose intersection is non-empty.

And the third one, a far reaching generalization of Helly's theorem, the famous (p, q)-theorem due to Alon and Kleitman, whose proof combined a large number of sophisticated tools and results that had been developed over the years since Helly's original theorem. It turns out that the fractional Helly theorem plays a crucial role in the proof of the (p, q)-theorem (one might even say the crucial role), furthermore there is a great variety of Helly-type theorems in mathematics, some of them, of a great relevance. For all of them, we may state their corresponding colorful and fractional versions.

2 New Trends in the Study of Helly and Tverberg Theorems

2.1 Helly and Tverberg Theorems Without Dimension

With the following research of Adiprasito, Bárány, Mustafa and Terpai [1], presented at this workshop, a new area in discrete geometry started, with the study of no-dimensional versions of classical theorems in convexity.

In the following results dimension plays an important role. What it initiated is the study of the dimension free versions of these theorems. These theorems are expected to be as useful as their classical versions.

For instance; In the Theorem of Carathéodory,

can one require here that $|Q| \leq r$ for some fixed $r \leq d$? The answer is obviously no. For instance when P is finite, the union of the convex hull of all r-element subsets of P has measure zero while $convP$ may have positive measure. So one should set a more modest target. One way to do this is to try to find, a subset $Q \subset P$ with $|Q| \leq r$ given $a \in convP$, so that a is close to $convQ$. This is the content of the following theorem:

Theorem 2.1 *Let P be a set of n points in $\mathbb{R}^d$, $r \in [n]$ and $a \in convP$. Then there exists a subset $Q \subset P$ with $|Q| = r$ such that*

$$d(a; convQ) \leq diamP/\sqrt{2r}.$$

When $r \geq d + 1$, the stronger conclusion $a \in convQ$ follows of course from Carathéodory's theorem. But in the statement of the theorem the dimension d has disappeared. So one can think of the n-element point set P as a set in $\mathbb{R}^n$ (or $\mathbb{R}^{n-1}$) with $a \in convP$. The conclusion is that for every $r < n$ the set P has a subset Q of size r whose convex hull is close to a. That is why the result is called "no-dimension Carathéodory theorem".

For the case of Helly's theorem, given a family of convex sets $K_1, \ldots K_n$ in $\mathbb{R}^d$, $n \geq d + 1$, define for $J \subset [n]$ define $K(J) = \cap_{j \in J} K_j$. What happens if the condition $K(J) \neq \emptyset$ only holds when $|J| = k$ and $k \leq d$? Then the statement fails to hold for instance when each K_i is a hyperplane (and they are in general position). But again, something can be saved, namely:

Theorem 2.2 *Assume $K_1, \ldots, K_n$ are convex sets in $\mathbb{R}^d$ and $k \in [n]$. If the Euclidean unit ball $B(b, 1)$ centered at $b \in \mathbb{R}^d$ intersects $K(J)$ for every $J \subset [n]$ with $|J| = d + 1$, then there is point $q \in \mathbb{R}^d$ such that*

$$d(q; Ki) < 1/\sqrt{k} \quad \textit{for all } i \in [n].$$

Again, the dimension of the underlying space is not important here. Note however that Helly's theorem is invariant under (non-degenerate) affine transformations while its dimension free version is not. The same applies to the no-dimension Carathéodory theorem. We mention that the condition $K(J) \cap B(b, 1) \neq \emptyset$ can be replaced by the following one: $K_i \subset B(b, 1)$ for all $i \in [n]$ and $K(J) \neq \emptyset$ for all $J \subset [n]$ with $|J| = k$.

The dimension free version of Tverberg's famous theorem is as follows.

Theorem 2.3 *Given a set P of n points in $\mathbb{R}^d$ and an integer $2 \leq k \leq n$, there exists a point $q \in \mathbb{R}^d$ and a partition of P into k sets $P_1, \ldots, P_k$ such that*

$$d(q, conP_i) \leq (2 + \sqrt{2})(\sqrt{k/n})diamP \quad \textit{for every } i \in [k].$$

2.2 New Colorful and Fractional Helly Results

Another important contribution shown at this workshop, was the idea of obtaining the fractional Helly's Theorem as a Corollary of the Colorful Helly's Theorem by using the following result of Holmsen in which large cliques in uniform hypergraphs with forbidden structures are obtained. The following definition generalizes the notion of induced $K_{2,m}$ in graphs, to complete m-tuple of missing edges in k-uniform hypergraphs.

Let $H = (V, E)$ be a k-uniform hypergraph. We call the set $M = \binom{V}{k} \setminus E$ the set of missing edges of H. Let $m \geq k$ an integer. A family $\tau_1, \dots, \tau_m \subset M$ is called a complete m-tuple of missing edges if

1. $\tau_i \cap \tau_j = \emptyset$ for all $i \neq j$, and
2. $\{t_1, \dots, t_m\}$ is a clique in H for all $t_i \in \tau_i$ and all $i \in [m]$.

For a k-uniform hypergraph H and $m \geq k$, let $c_m(H)$ denote the number of cliques on m vertices in H. We may now state the following result (see [8]).

Theorem 2.4 Holmsen *For any $m \geq k \geq 2$ and $\alpha \in (0, 1)$, there exists a constant $\beta = \beta(\alpha, k, m) > 0$ with the following property: Let H be a k-uniform hypergraph on n vertices and $c_m(H) \geq \alpha\binom{n}{m}$. If H does not contain a complete m-tuple of missing edges, then $\omega(H) \geq \beta n$.*

Here $\omega(H)$ denotes the maximum number of vertices in a clique contained in H. This result generalize the following classical theorem in extremal graph theory.

Theorem 2.5 Gyárfás-Hubenko-Solymosi *Let G be a graph on n vertices and at least $\alpha\binom{n}{2}$ edges. If G does not contain $K_{2,2}$ as an induced subgraph, then $\omega(G) \geq \frac{\alpha^2}{10} n$.*

Proof of the fractional Helly theorem from the colorful Helly theorem using this technique.
Define a $(d+1)$-uniform hypergraph $H = (F, E)$ where $E = \{\sigma \in \binom{F}{d+1} \mid \cap_{K \in \sigma} \neq \emptyset\}$. By hypothesis, H has at least $\alpha\binom{n}{d+1}$ edges, and by the Colorful Helly Theorem H does not contain a complete $(d+1)$-tuple of missing edges. So by Holsem's Theorem, with $k = m = d + 1$, there exists a $\beta > 0$ such that H has a clique on βn vertices. By Helly's theorem, the members of F contained in this clique have non empty intersection.

It is a not a surprise now, due to Holmsen Theorem 2.4, that in all of them, *"the fractional version can be derived as a purely combinatorial consequence of the colorful version"*.

In other words there are various fractional Helly theorems and it is of considerable interest to understand what conditions can be imposed on a set system which guarantees that it admits the "fractional Helly property". As we will see, this can be understood via Holmsen Theorem 2.4 in which large cliques in hypergraphs with forbidden structures are obtained.

One example, which is of great interest, is the topological generalization of the Helly's Theorem. Kalai's proof of the Topological Fractional Helly relies on his technique of algebraic shifting, but here it is obtained of from Holmsen's Theorem 1.6 and the colorful Helly theorem due to Kalai and Meshulam

Another question of Gil Kalai is whether "Radon implies fractional Helly"? We now describe an axiomatic setting in which the above question can be made precise. A convexity space is a pair (X, C) where X is a (non-empty) set and C is a family of subsets of X satisfying the following properties:

- $\emptyset, X \in C$.
- $A, B \in C$ implies $A \cap B \in C$

Given a convexity space (X, C) and a subset $Y \subset X$ we define the convex hull of Y, denoted by conv(Y), to be the intersection of all the convex sets containing Y. Define the Radon number of (X, C) as the smallest integer r_2 (if it exists) such that every subset $P \subset X$ with $|P| \geq r_2$ can be partitioned into two parts P_1 and P_2 such that conv$(P_1)\cap$ conv$(P_2) \neq \emptyset$. The following relevant result is a fractional Helly theorem for general convexity spaces with bounded Radon number answering Kalai's question (see [9]).

Theorem 2.6 Holmsen and Lee *For every $r \geq 3$ and $\alpha \in (0, 1)$ there exists an $m = m(r)$ and a $\beta = \beta(\alpha, r) \in (0, 1)$ with the following property: Let F be a family of $n \geq m$ convex sets in a convexity space with Radon number at most r. If at least $\alpha\binom{n}{m}$ of the m-tuples of F have non-empty intersection, then there are at least βn members of F whose intersection is non-empty.*

Squetch of the Proof First establish a colorful Helly theorem for general convexity spaces, and this is where the integer $m(r)$ appears as the number of colors needed. Next, consider the 'intersection hypergraph" carrying the information of which subfamilies of F are intersecting. The colorful Helly theorem may then be interpreted as forbidding certain patterns from the intersection hypergraph. We can then apply Holmsen Theorem 1.6 concerning the clique number of dense uniform hypergraphs with forbidden substructures.

Another deep consequence of this ideas is that a weak ϵ-net theorem for convexity spaces with a bounded Radon number can be obtained. This answers a question of Bukh and extends a recent result of Moran and Yehudayoff.

Radon's theorem is one of the cornerstones of convex geometry. It implies many of the key results in the area such as Helly's theorem. As described above, the work of Andreas Holmsen and Dong-Gyu Lee, Rado's Theorem implies a fractional Helly's theorem together with a colorful strengthening of Helly's theorem. Consequently, this yields an existence of weak epsilon nets and a (p, q)-theorem.

P. Patak and Z. Patakova can obtain these results even without assuming convexity, replacing it with very weak topological conditions. Moreover, using the recent result of P. Patak and Gil Kalai, it is possible to bring the fractional Helly number for open sets in the plane or on a surface down to three. This also settles a conjecture of Andreas Holmsen, Minki Kim, and Seunghun Lee about an existence of a (p, q)-theorem for open subsets of a surface.

In a recent work, Zuzana Patakova has shown at the workshop that for a finite family $\mathcal{F}$ of sets in $\mathbb{R}^d$, one can use Betti numbers of intersections of subfamilies of $\mathcal{F}$, to bound the Radon's number of $\mathcal{F}$. The result has interesting consequences, some of them are easy or standard, other follow from a result of Holmsen and Lee. Let us name just few: variants of Helly's, Tverberg's, colorful and fractional Helly theorems, existence of weak ε-nets, (p, q)-theorems. Nevertheless, the original bounds on Radon's number are too large to be widely applicable.

2.3 Quantitative Helly's Theorems

Many generalizations and extensions of Helly's and Tverberg's theorems have been proven lately, including colorful, topological, and integer versions for both theorems. A particular family of generalizations of both theorems, called the quantitative versions, gives conditions that guarantee that the intersection of a family of convex sets in $\mathbb{R}^d$ is large. For example, we can ask for bounds on the volume of the intersection of a family of convex sets.

Theorem 2.7 (Bárány, Katchalski, Pach). *Let F be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every $2d$ or fewer sets in F has volume at least one, then the volume of $\cap_F$ is at least d^{-2d^2}.*

One can easily show that we cannot expect to conclude that the volume of $\cap_F$ is at least one if $d \geq 2$, so there is no exact Helly theorem for the volume. The lower bound for the volume of $\cap_F$ has been improved recently. First by Naszódi, giving a bound of $O(d^{-2d})$ and then by Brazitikos, giving a bound of $O(d^{-3d/2})$. If we know that the intersection of subfamilies of larger cardinality, d for some constant, have volume greater than or equal to one, Brazitikos showed that we can get a lower bound of $O(d^{-d})$ for the volume of $\cap_F$. If one is willing to check much larger subfamilies, it was shown that we can get a bound of $1 - \epsilon$ on the volume of $\cap_F$ if we know that the intersection of every $\Theta(\epsilon^{-(d-1)/2})$ sets has volume at least one.

Quantitative Tverberg theorems are much more recent, and there are several interpretation of what the correct version should be. The results we consider here, due to Damásdi, Földvári and Soberon are versions in which points are replaced by convex sets. They seek a partition of the family such that the intersection of the convex hulls of the parts is large. As an analogue for the Bárány-Katchalski-Pach theorem it is presented the following result. The number of sets needed can be reduced slightly if r is a prime power.

Theorem 2.8 Tverberg for volume *Let r, d be positive integers and F be a family of $(r-1)\left(\frac{d(d+3)}{2}+1\right)+1$ sets of volume one in $\mathbb{R}^d$. Then, there exists a partition of F into r parts $A_1, \dots, A_r$ such that the volume of $\cap_1^r conv(A_j)$ is at least d^{-d}.*

Quantitative Helly theorems can be considered as a bridge between combinatorial geometry and analytic convex geometry. The results of Naszódi and Brazitikos show how they are related to the sparsification of John decompositions of the identity. Moreover, the results of De Loera, La Haye, Rolnick, and Soberon show

how they are related to the theory of approximation of convex sets by polytopes Furthermore, the results of Rolnick and Soberon show how the colorful versions are related to the analytic properties of floating bodies. Some of the results use the topological versions of Helly's theorem and of Tverberg's theorem in their proofs. Until the work of Damásdi, Földvári and Soberon, topological methods have not been used before for quantitative variations. The results depend on two main components: the function we work with, and the family of sets we use to witness that we achieve a desired value in the function. The Helly numbers (i.e., the size of the subfamilies we must check) in our results are determined by the dimension of the space of possible witness sets, and they are often optimal. The Tverberg theorems have a similar dependence. The fact that the space of convex sets in $\mathbb{R}^d$ has in infinite dimension gives an intuitive idea of why the loss of volume is unavoidable in the Bárány-Katchalski-Pach theorem. It's important to note that just finding good families of witness sets is not enough. Otherwise, we would be able to obtain exact quantitative results for the diameter, as it is always realized by a segment. John's theorem implies the following theorem.

Theorem 2.9 Quantitative Helly's Theorem with ellipsoids as witness *Let C be a finite family of convex bodies in $\mathbb{R}^d$, and assume that for any choice $C_1, \ldots, C_{\frac{d(d+3)}{2}} \in C$, the intersection $\bigcap C_i$ contains an ellipsoid of volume 1. Then $\cap_C$ also contains an ellipsoid of volume 1.*

As we know from Section 1.2, Lovász proved a colorful version of Helly's theorem and later Bárány gave another proof that uses his colorful version of Carathéodory's Theorem. In the same vein, it is possible to prove a colorful version of Theorem 1.11.

Theorem 2.10 *Let $C_1, \ldots, C_{\frac{d(d+3)}{2}}$ be finite families of convex bodies in $\mathbb{R}^d$, and assume that for any colorful selection $A_1 \in C_1, \ldots, A_{\frac{d(d+3)}{2}} \in C_{\frac{d(d+3)}{2}}$ the intersection $\cap A_i$ contains an ellipsoid of volume 1. Then for some j, an ellipsoid of volume 1 is contained in $\cap_{C_j}$.*

Sarkar, Xue and Soberon, using matroids, recently obtained a result concerning volume with the number of selected sets being 2d (see [14]).

Theorem 2.11 Sarkar, Xue and Soberon *Let $C_1, \ldots, C_{\frac{d(d+3)}{2}}$ be finite families of convex bodies in $\mathbb{R}^d$, and assume that for any colorful choice of $2d$ sets, $A_{i_k} \in C_{i_k}$ for each $1 \leq k \leq 2d$ with $1 \leq i_1 < \cdots < i_{2d} \leq d(d+3)/2$, the intersection $\cap_1^{2d} A_{i_k}$ has volume at least 1. Then, there exists an $1 \leq i \leq d(d+3)/2$ such that $\cap_{C_i}$ has volume at least $d^{-O(d)}$.*

The smaller the number of color classes in a Colorful Helly-type theorem, the stronger the theorem is, so the main result is that instead of $O(d^2)$ color classes, $3d$ are sufficient

Theorem 2.12 Damásdi, Földvári and Naszódi *Let $C_1, \ldots, C_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$, and assume that for any colorful choice of $2d$ sets, $A_{i_k} \in C_{i_k}$ for each $1 \leq k \leq 2d$ with $1 \leq i_1 < \cdots < i_{2d} \leq 3d$, the intersection $\cap_1^{2d} A_{i_k}$ contains an ellipsoid of volume at least 1. Then, there exists an $1 \leq i \leq 3d$ such that $\cap_{C_i}$ contains an ellipsoid of volume at least $c^{d^2} d^{5d^2/2}$, with some universal constant $c \geq 0$.*

2.4 Helly numbers for discrete sets, crystals, cut-and-project sets

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Doignon's Theorem also known as Doignon-Bell-Scarf theorem, says that a finite family of convex sets in $\mathbb{R}^d$ intersect at a point of $\mathbb{Z}^d$ if every 2^d of them intersect at a point of $\mathbb{Z}^d$. Later Averkov and Weismantel gave a mixed version of Helly's and Doignon's theorems which includes them both, this time the intersection of the convex sets is required to be in $\mathbb{Z}^{d-k} \times \mathbb{R}^k$ and this can be guaranteed if every $2^{d-k}(k+1)$ sets intersect in such a point.There are also interesting generalization given by De Loera, La Haye, Oliveros and Roldán-Pensado with some other fixed semi-algebraic groups, and the corresponding fractional version of Doignon's theorem was proved by Bárány and Matousek. At this workshop A. Garber [7] obtained that Helly-type theorems with finite Helly numbers exist for periodic and certain quasiperiodic sets in Euclidean space of any dimension, though the bounds in the latter case seem to be extremely non-optimal. Garber also probed that for a wider class of Meyer sets Helly numbers could be infinite.

Helly's theorem also led to the study of covering numbers (also sometimes called piercing numbers or hitting numbers) of families of sets, that is, the minimal number of points needed to "pierce" every set in a family given some local intersection property. Helly's theorem is sharp in general, but improved bounds on covering numbers can be obtained if one restricts the convex sets in question. Indeed, extensive work has been done on the covering numbers of families of disks, boxes, line segments, homothets of centrally symmetric bodies, and other convex sets As we mention with Doignon's theorem, determining covering numbers in discrete arrangements is often harder than in the continuous case, as the covering sets of points are required to lie in the lattice as well.

In this context it was presented at the worksop the work of D. Oliveros, C. O'Neill and S. Zerbib (see [12]. on the geometry and combinatorics of discrete line segment hypergraphs, where they study the covering numbers of r-segment hypergraphs where a r-segment hypergraph H is a hypergraph whose edges consist of r consecutive integer points on line segments in the integer lattice of the plane, as well as some recent bounds on the chromatic numbers.

2.5 Data-Clasification Complexes and Tverberg Theorem

Motivated by questions from the performance of data classification algorithms, such as multi-class logistic regression method, De Loera, Hogan, Oliveros and Yang [5] presented other versions of Tverberg's theorem where the nerve complex describing the intersections of classes is not a simplex.

To state the results precisely, let us start with some terminology and notation typical of geometric topological combinatorics. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of convex sets in $\mathbb{R}^d$. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex with vertex set $[m] := \{1, 2 \dots, m\}$ whose faces are $I \subset [m]$ such that $\cap_{i \in I} F_i \neq \emptyset$.

Given a collection of points $S \subset \mathbb{R}^d$ and an n-partition into n color classes $\mathcal{P} = S_1, \dots, S_n$ of S, *the nerve of the partition*, $\mathcal{N}(\mathcal{P})$ is defined to be the nerve complex $\mathcal{N}(\{conv(S_1), \dots, conv(S_n)\})$, where $conv(S_i)$ is the convex hull of the elements in the color class i. Similarly, given a partition $\mathcal{P}$, the *intersection graph of the partition*, denoted $\mathcal{N}^1(\mathcal{P})$, denotes the 1-skeleton of the nerve of $\mathcal{P}$.

Given a simplicial complex K, and a finite set of points S in $\mathbb{R}^d$, it is sad that K is *partition induced on* S if there exists a partition $\mathcal{P}$ of S such that the nerve of the partition is isomorphic to K. We say that K is d*-partition induced* if there exists at least one set of points $S \subset \mathbb{R}^d$ such that K is partition induced on S. It was shown by G. Y. Perelman that every d-dimensional simplicial complex is $(2d+1)$-partition induced on some point set. This result is in fact optimal, because the barycentric subdivision of the d-skeleton of a $(2d+2)$-dimensional simplex is not $2d$-partition induced.

A simplicial complex K is sad to be d*-Tverberg* if there exists a constant $Tv(K, d)$ such that K is partition induced on all point sets $S \subset \mathbb{R}^d$ in general position with $|S| > Tv(K, d)$. The minimal such constant $Tv(K, d)$ is called the *Tverberg number* for K in dimension d. Note that it is possible to re-state the classical Tverberg's theorem as follows:

Theorem 2.13 Tverberg's theorem rephrased *The* $(m-1)$*-simplex is a* d*-Tverberg complex for all* $d \geq 1$*, with Tverberg number* $(d+1)(m-1)+1$.

Recall that the k*-hypergraph Ramsey number* $R_k(m)$ is the least integer N such that every red-blue 2-coloring of all k-subsets of an N-element set contains either a red set of size m or a blue set of size m, where a set is called red (blue) if all k-subsets from this set are red (or respectively blue).

Then, in this workshop Jesus De Loera presented the following generalization of the classical Tverberg's theorem by showing that similar theorems exist where other simplicial complexes -not just simplices- appear as the nerve of the partition.

Theorem 2.14 *All trees and cycles are* d*-Tverberg complexes for all* $d \geq 2$.

(A) Every tree T_n *on* n *nodes, is a* d*-Tverberg complex for* $d \geq 2$*. The Tverberg number* $Tv(T_n, d)$ *exists and it is at most* $R_{d+1}((d+1)(n-1)+1)$*. More strongly,* $Tv(T_n, 2)$ *is at most* $\binom{4n-4}{2n-2} + 1$.

(B) Every n*-cycle* C_n *with* $n \geq 4$ *is a* d*-Tverberg complex for* $d \geq 2$*. The Tverberg number exists and* $Tv(C_n, d)$ *is at most* $nd + n + 4d$.

The proof of Theorem 2.14 relies on several powerful non-constructive tools such as the Ham-Sandwich theorem, a characterization of oriented matroids of cyclic polytopes, and the multi-dimensional version of Erdös-Szekeres theorem. These tools are enough to show the existence of a Tverberg number $Tv(T_n, d)$, but the bounds are far from tight.

2.6 Helly Theorems and Embeddings

The study of embeddings of k-complexes into $2k$-manifolds has deep consequences in the problem of finding bounds for the Helly's number in a certain Helly-type theorems, where the ambient space is a (suitable) manifold. Indeed, modulo a recent Lefschetz theorem of Adiprasito, these bounds are consequence of the research of M. Tancer and P. Patak [13], in which they prove the existence and completeness of obstructions for some kind of suitable embeddings.

Let K be a simplicial k-complex and M be a closed PL $2k$-manifold. The first aim is to define an obstruction for embedability of K into M via the intersection form on M. For description of the obstruction, it is needed a technical condition which is satisfied, in particular, either if M is $(k-1)$-connected or if K is the k-skeleton of n-simplex, for some n. Under the technical condition, if K (almost) embeds in M, then the obstruction vanishes. In addition, if M is $(k-1)$-connected and $k \geq 3$, then the obstruction is complete, that is, it is possible to get the reverse implication.

3 Intersection Patterns

The study of the intersection patterns of convex sets is a substantial part of combinatorial geometry. Given an intersection pattern of arbitrary sets in Euclidean space, is there an arrangement of convex open sets in Euclidean space that exhibits the same intersections? This question is combinatorial and topological in nature but it is motivated in neuroscience, specifically from the study of neurons called place cells. The discovery of place cells by O'Keefe et al. in 1971 was a major breakthrough that led to a shared 2014 Nobel Prize in Medicine or Physiology. A place cell encodes spatial information about an organism's surroundings by firing precisely when the organism is in the corresponding place field. In this context, a codeword represents the neural firing pattern that occurs when the organism is in the corresponding region of its environment: the ith coordinate is 1 if and only if the organism is in the place field of neuron i. The resulting set of codewords is called a neural code. Place fields can be modeled by convex open sets, so we are interested in the following restatement of the question that opened this work: Which neural codes can arise from a collection of convex open sets? To address this problem, Giusti and Itskov identified a local obstruction, defined via the topology of a code's simplicial complex, and proved that convex neural codes have no local obstructions. Codes without local obstructions are called locally good, as the obstruction prevents the code from encoding the intersections of open sets that form a good cover (for instance, if the sets are convex). If such a good cover exists (for instance, from a collection of convex open sets), then the code is a good-cover code. Thus:

C is convex $=>$ C is a good-cover code $=>$ C is locally good.

The converse of the first implication is false. The second implication is the starting point of the research of Chen, Flick and Shiu (see [4]). They prove that the implication is in fact an equivalence: every locally good code is a good-cover code. They also prove that the good-cover decision problem is undecidable. Next, they discover a new, stronger type of local obstruction that precludes a code from being convex. Like the prior obstruction, the new obstruction is defined in terms of a code's simplicial complex, but in this case the link of "missing" codewords must be "collapsible" (which is implied by "contractible", the condition in the original type of obstruction). They call codes without the new obstruction locally great, and examine the corresponding decision problem. They also prove that the locally-great decidability problem is decidable, and in fact NP-hard. Thus, these results refine the implications we saw earlier, as follows:

C is convex $=>$ C is locally great $=>$ C is a good-cover code $=>$ C is locally good.

Finally, they add another implication to the end of those listed above, by noting that every locally good code can be realized by connected open sets, but not vice-versa. Taken together, these results resolve fundamental questions in the theory of convex neural codes.

Janos Pach is also interested in geometric intersection patterns and the theory of topological graphs. The intersection graph of a set system S is a graph on the vertex set S, in which two vertices are connected by an edge if and only if the corresponding sets have nonempty intersection. It was shown by Tietze (1905) that every finite graph is the intersection graph of 3-dimensional convex polytopes. The analogous statement is false in any fixed dimension if the polytopes are allowed to have only a bounded number of faces or are replaced by simple geometric objects that can be described in terms of a bounded number of real parameters. Intersection graphs of various classes of geometric objects, even in the plane, have interesting structural and extremal properties. Many of the questions discussed by Pach were originally raised by Berge, Erdös, Grünbaum, Hadwiger, Turán, and others in the context of classical topology, graph theory, and combinatorics (related, e.g., to Helly?s theorem, Ramsey theory, perfect graphs). The rapid development of computational geometry and graph drawing algorithms in the last couple of decades gave further impetus to research in this field. A topological graph is a graph drawn in the plane so that its vertices are represented by points and its edges by possibly intersecting simple continuous curves connecting the corresponding point pairs. J. Pach give applications of the results concerning intersection patterns in the theory of topological graphs.

4 Another related results in discrete geometry

Consider a situation when n players want to divide a "continuous" in certain sense resource X among themselves. We assume that, for each partition of X into n pieces (some possibly empty), each player would be satisfied to take one of the partition pieces, the choice of a player need not be unique. When no player prefers an empty piece of the resource, the existence of an equilibrium, where every player receives one piece of the partition and is satisfied, is guaranteed by Gale's theorem. For such situations, when every player receives what she/he prefers from a given partition, the term envy-free partition is usually used.

Making one step from the classical situations, in this research Avvakumov and Karasev make a generalization, following Meunier and Zerbib work. The resource might come with some cost, so it might naturally happen that for certain partitions the cost of all the non-empty pieces is too high for a player. Then some of the players might prefer to take an empty piece. As in Gale's theorem and other classical results, they make a natural assumption on player's preferences, mathematically speaking, a player prefers a part if in another, but arbitrarily close to given partition configuration she/he also prefers this part.

Avvakumov and Karasev [2] will mostly have in mind the segment partition problem, for a unit interval $[0;1]$, they consider its partitions into n closed (possibly empty) segments with pairwise disjoint interiors. As a simple example, every player may rate the parts with her/his own integrable "value" function f_i on $[0;1]$, and prefers any of those segments which maximize the value of the integral of f_i over them. Following the classical works, they consider a more general setting than the "value" function; they allow any player to rate the pieces of a given partition with more complicated logic. The very term "envy-free partition" is motivated by the fact that a player's preference of a certain piece may depend on how the rest of the resource is partitioned, and in the solution for the problem no player has envy to take a different piece than she/he is given.

In the special case of the segment partitioning problem of Segal-Halevi, it was proved that envy-free segment partitions exist for $n = 3$ (the case $n = 2$ is an easy exercise). In a pervious work of Meunier and Zerbib, the result was extended to $n = 4$, or any prime n. In this research Avvakumov and Karasev give a complete solution to the problem in this setting. They prove that if n is a prime power then an envy-free segment partitioning always exists. Conversely, for every n which is not a prime power, there exists an instance of the segment envy-free partition problem with no solution.

The convex dimension of a k-uniform hypergraph is the smallest dimension d for which there is an injective mapping of its vertices into $\mathbb{R}^d$ such that the set of k-barycenters of all hyperedges is in convex position. L. Martnez-Sandoval and Arnau Padrol [11] completely determine the convex dimension of complete k-uniform hypergraphs, which settles an open question by Halman, Onn and Rothblum, who solved the problem for complete graphs. They also provide lower and upper bounds for the extremal problem of estimating the maximal number of hyperedges of k-uniform hypergraphs on n vertices with convex dimension d. Their results have direct interpretations in terms of k-sets and (i, j)-partitions, and are closely related to the problem of finding large convexly independent subsets in Minkowski sums of k point sets.

Emo Welzl and Uli Wagner obtained a new and relevant approach to bistellar and edge Flip Graphs of triangulations in the plane for the study of its geometry and connectivity. The set of all triangulations of a finite point set in the plane attains structure via flips: The graph, where two triangulations are adjacent if one can be obtained from the other by an elementary flip. This is an edge flip for full triangulations, or a bistellar flip for partial triangulations (where non-extreme points can be skipped). It is well-known (Lawson, 1972) that both, the edge flip graph and the bistellar flip graph are connected. For n, the number of points in general position assumed, Welzl and Wagner showed that the edge flip graph is (n/2 - 2)-connected and the bistellar flip graph is (n-3)-connected. Both bounds are tight. This matches the situation for regular triangulations (a subset of the partial triangulations), where (n-3)-connectivity was known through the secondary polytope (Gelfand, Kapranov, Zelevinsky, 1990) and Balinski's Theorem. They show that the edge flip graph can be covered by 1-skeletons of polytopes of dimension at least n/2-2 (products of associahedra). Similarly, the bistellar flip graph can be covered by 1-skeletons of polytopes of dimension at least n-3 (products of secondary polytopes).

The scissors congruence conjecture for the unimodular group is an analogue of Hilbert's third problem for the equidecomposability of polytopes. Ramiréz-Alfonsin, Fernandes, Pina and Robins gave a proof of this conjecture for polytopes naturally associated to graphs whose vertices have degree one or three. The key ingredient in the proof is the nearest neighbor interchange on graphs and a naturally arising piecewise unimodular transformation. They also studied Ehrhart quasi-polynomials results for this class of polytopes.

Oleg Musin and Andrey Malyutin [10] introduced and studied a new class of extensions for the Borsuk–Ulam theorem. His approach is based on the theory of Voronoi diagrams and Delaunay triangulations. One of his main results is as follows.

Theorem 4.1 A. Malyutin and O. Musin *Let S^m be a unit sphere in $\mathbb{R}^{m+1}$ and let $f : S^m \to \mathbb{R}^n$ be a continuous map. Then there are points p and q in S^m such that*

- $\|p - q\| \geq \sqrt{2 \cdot \frac{m+2}{m+1}}$;
- *$f(p)$ and $f(q)$ lie on the boundary of a closed metric ball B in R^n whose interior does not meet $f(S^m)$.*

5 Developments in Geometry and Convex Geometry

The following is known as the geometric hypothesis of Banach: let V be an m-dimensional Banach space with unit ball B and suppose that for an integer $n > 1$ all n-dimensional subspaces of V are isometric (all the n-sections of B are affinely equivalent). In 1932, Banach conjectured that under this hypothesis V is a Hilbert space (the boundary of B is an ellipsoid). Gromow proved in 1967 that the conjecture is true for $n =$even and Dvoretzky derived the same conclusion under the hypothesis $n =$infinity. In this workshop L. Montejano [3] show the joint work with Bor, Hernández and Jimenénez where they prove this conjecture for all positive integers of the form $n = 4k + 1$, with the possible exception of 133. The ingredients of the proof are classical homotopic theory, irreducible representations of the orthogonal group and convex geometry. Indeed, the proof relies on a new characterization of ellipsoids in $\mathbb{R}^n$, $n \geq 5$, as the only convex bodies all of whose linear hyperplane sections are linearly equivalent affine bodies of revolution.

In analogous directions, we have the work in characterizations of spheres by E. Morales, in F-convexity by Liping Yaun and in projective geometry by J. Bracho.

6 Conclusions

The workshop was successful in all posible ways, bringing together old and new colleagues from all over the world. We had participants from many countries such as Russia, Germany, France, USA, Mexico, Korea, Check Republic, Hungary and China, among others. The talks were far from being the only academic activity of the workshop. We had many formal and informal mathematical discussions and all these activities have given rise to many new research projects and new collaboration.

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