Topological Phases of Interacting Quantum Systems

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1 Overview of the Field

Topological insulators are semiconductors with a Fermi level lying in a mobility gap of the bulk material, which nevertheless have non-trivial topology in the Bloch bands (Chern numbers and higher winding numbers). Via a bulk-boundary correspondence this non-trivial topology leads to conducting surface states that are not susceptible to Anderson localization. The presence of such surface states is often also used as the defining characteristic of a topological insulator. Since the early theoretical proposals [49, 77] the theory has now reached some maturity from a theoretical physics perspective [55, 81, 79], and also the bulk-boundary correspondence is rather well understood [45, 46, 77, 25], even though for some systems such as the quantum spin Hall effect the situation is not settled. Also the effect of further symmetries and defects in topological insulators has been analyzed [29, 87, 48]. Numerical methods have been developed to calculate the topological invariants also for disordered systems [68, 72, 42, 57]. There are good reviews [44, 13, 31], and a growing list of materials that actually are topological insulators [2]. The theory also has been transposed to other wave equations, such as driven Floquet systems [78], photonic crystals [40], bosonic systems [85], matter waves [71]. An issue that is still under investigation, even from a theoretical physics perspective, is the role of interactions both for new effects (such as the fractional quantum Hall effect) or the stability of the above mentioned topology to weak interactions. Similar issues appear for the classical waves, when one passes from linear to non-linear regimes. There are only some isolated results in higher dimension, but in the understanding of interacting one-dimensional systems there has been considerable progress [29, 89, 10], mainly based on matrix product states [28].

The grand picture of the field concerned with rigorous analysis of topological effects in condensed matter systems consists at this moment of several quite disconnected pieces:[1] 1) The aperiodic non-interacting condensed matter systems are quite well understood. Rigorous results exist for both bulk and bulk-boundary programs for contexts as general as disordered, quasi-periodic, quasi-crystalline and amorphous systems. This level of understanding enabled engineering of new topological materials and meta-materials. 2) Topological order is a concept introduced and championed by theoretical physicists: A physical system is said to display topological order if it can be formulated on triangulations of arbitrary genus surfaces, it manifests spectral degeneracy which grows exponentially with the genus of the surface and its low-energy excitations possess non-trivial self-statistics. The data associated to these theoretical models are naturally formulated in terms of tensor categories but a comprehensive representation theory of these categories is lacking. As such, most of these models remain rather abstract, with little and sometimes no connection with the physical condensed matter systems. Furthermore, the theoretical models are finite and it is not clear how to coherently define a thermodynamic limit for them. 3) There are several proposals, coming from the community of theoretical physicists, of topological invariants for correlated periodic topological insulators. However, almost all these invariants loose their meaning when periodicity is not present. Hence a key challenge is how to proceed in regimes where both correlations and disorder are equally strong. 4) Using methods coming from constructive field theory and traditional many-body physics, such as Ward identities, re-normalization techniques, Lieb-Schultz-Mattis theorem, modern forms of adiabatic theorem, etc., there has been limited, but nevertheless exciting and extremely important progress on defining topological invariants for correlated systems under conditions of periodicity or week disorder.

Triggered by all these challenges and guided by some of the advances, there has been a considerable effort in the mathematical physics community to develop clear concepts and to supply rigorous proofs, as well as to place and formulate the entire effort into specific frameworks of modern mathematics. Below, we mention some of the important mathematical results obtained before 2017, roughly, which more or less served as starting points for most of the discussions we had at Oaxaca. In the next sections, we review more recent developments and specify how they integrated with the program of our workshop.

2 Early Developments

First of all, in the 1980's Jean Bellissard [11] identified C^*-algebras to be the natural framework to formalize and analyze aperiodic but homogeneous condensed matter systems.[2] Let us recall that this development happened at times when von Neumann algebras were dominating the discussions in the mathematical physics community, due to their effective use in the constructive field theory program [36]. In his work, Bellissard states several fundamental reasons why C^*-algebras should be used: 1) The algebraic structures of C^*-algebras determine uniquely their topology because the norm of an element can be expressed in terms of its spectral radius and the latter is a purely algebraic concept; 2) As opposed to

[1] References will be supplied in the following sections, where all these points will be elaborated.

[2] Due to personal reasons, Jean had to cancel his trip to Oaxaca. We wish him well.

von Neumann algebras, the (separable) C^*-algebras display countable K-groups and, as such, K-theory can be transformed into an effective tool in spectral theory; 3) The analysis can be formulated directly in the thermodynamic limit. For example, a theory of charge transport can be cleanly developed without appealing to finite-volume or rational magnetic flux approximations [82], as usually done with other formalisms.

A second stepping stone was set in place when index theory was identified in [12] as the natural framework for the analysis of the quantized response coefficients in the strong disorder regime where the spectral gaps are replaced by mobility gaps. The principle discovered in this work is that the index theorems, which are usually formulated on certain sub-algebras of smooth elements, can be pushed over certain non-commutative Sobolev spaces which cover the mobility gap regime. To date, this principle remains the only way to demonstrate the stability of the topological invariants beyond the spectral gap regime.

A third major development came from [52], where the connecting maps of the K-theory were identified as the engine of the bulk-boundary correspondence principle. More specifically, this work showed that the algebras of boundary, half-space and bulk physical observables enter into a long exact sequence, which leads to a 6-term exact sequence between the K-groups. The connecting maps then act like elevators between bulk and boundary phenomena. In particular, they can detect when a physical boundary induces topological boundary spectrum which fills the bulk spectral gaps and cannot be removed by bulk deformations or by changing the boundary conditions. This phenomenon is known as the "spectral statement" of the bulk-boundary principle.

To be complete, the bulk-boundary principle must also contain a "dynamical statement" which asserts that the boundary modes diffuse even in the presence of large boundary disorder. A major development was the discovery in [74] that the index theorems resulting from the pairing between K-theory and cyclic cohomology, as applied to the boundary algebra, can supply a natural and rigorous proof of the dynamical statement.

The four elements mentioned above remain the only model known to us for establishing the bulk-boundary principle for a condensed matter system. These early works, however, have been expanded and applied to many other contexts, notably to include fundamental symmetries of topological insulators and superconductors [12, 32, 73, 74, 39, 88, 50], and to define the $\mathbb{Z}_2$-invariants [4, 83, 30]. Parallel to that, a homotopy classification of Bloch vector bundles with symmetries has been established [24, 53]. The bulk-boundary correspondence has been partially re-formulated using T-duality [63] (there is no dynamical statement in this work). A very important new trend is the migration towards the more general framework of Kasparov's K-theory. This started with a formulation of the bulk-boundary principle using Kasparov's product [16, 17] fallowed by derivations of generalized index theorems using KK-theory [75]. Over many years, there has been an ongoing effort to understand the stability of phases in quantum spin systems [15, 9], and more recently this has been used also in topological spin systems [23, 8]. In another direction, field theoretic methods (Ward identities) have allowed to show that conductances are quantized for periodic interacting fermionic many body systems [33, 34].

3 Recent Results on Index Theory of Non-Interacting Systems

While the main object of the meeting was to present and discuss advances in the field of interacting topological systems, there are presently still numerous original contributions to the field of non-interacting systems. One of the reasons for this is that there is a still growing interest to engineer topological systems and to use their physical properties in practical applications. This always involves mathematical analysis before experimental realizations can be attacked. Hence the robust mathematical concepts known to hold for non-interacting fermionic systems are transposed to other situations. There are also still numerous open questions in the framework of non-interacting systems that are of intrinsic mathematical interest. During the meeting there were a number of presentations which in a wider sense fit into this category. Many colleagues work in parallel on such novel situations as well as on interacting systems where more conceptual problems have to be faced.

First let us describe the recent developments on K-theory and index theory. Of general interest is a new technique to compute $\mathbb{Z}$ and $\mathbb{Z}_2$-invariants numerically via the so-called spectral localizer which is a new type of Dirac operator that comprised both the K-theoretical and K-homological information. This tool was suggested in concrete situations by Loring [57] and recently it was shown to indeed be connected to standard invariants in the very broad set-up of index pairings [58, 59] (talk by Loring). A new proof based on spectral flow has also been found [60]. One interesting open question concerns a KK-theoretic interpretation of the spectral localizer. Indeed, it looks like a Kasparov product, albeit not of a standard type. Another question is how the spectral localizer can be used for the calculation of weak invariants. Let us also mention another recent result that may be of broader interest for index theory. It has been shown that the insertion of non-abelian monopoles leads to a spectral flow that is equal to the strong topological invariants [21].

Other works aim to identify the suitable C^*-algebras for the physical description of a given system. One recent proposal shows how to use tools from coarse geometry to construct rather large algebras that merely distinguish systems with differing strong invariant [27] (talk by Meyer). Other works construct groupoid C^*-algebras for the description of aperiodic lattices and amorphous systems and then carry out index calculations in that framework [19, 18] (talk by Mesland). The KK-theoretic approach to the bulk-boundary correspondence has been further developed [3] (talk by Max). On another page, for one-dimensional systems with quasiperiodic potentials of Sturmian Kohmoto-type a very careful construction of the bulk-boundary exact sequence is needed and this allows to understand the boundary states in such systems [51] (talk by Kellendonk). Yet other exact sequences of Toeplitz type are needed to show the existence of corner states [47] (talk by Hayashi). Also for (driven) Floquet systems the K-theoretic [80] as well as analytic [38, 86] approach have been used successfully to understand the nature of boundary states (talk by Graf). This also allowed (finally after so many years) to understand the surface states in topological quantum walks such as the Chalker-Coddington model [80]. Finally, there is still an ongoing effort to understand the bulk-boundary correspondence in the mobility gap regime. There has been progress on one-dimensional chiral systems in this respect [37] (talk by Graf), but several questions on the higher dimensional cases remain open.

The construction of smooth Wannier functions is a classical objective of periodic solid state physics. It has recently been shown that such functions only exist when the K-theory invariants are trivial [67] (talk by Panati). This can be extended to a very general context of non-commutative Bloch theory [61] (talk by Thiang). Another recent theme of the physics literature concerns Weyl semimetals. While several elements of such models have features in common with topological insulators, namely Weyl semimetals can be understood as being transition points between different insulators, it has not been clear in how far such systems are stable under random perturbations or interactions, nor whether the bulk-boundary correspondence transposes. Partial progress has been made on these issues. An extension of the algebraic formalism to disordered Weyl semimetals has been developed and this allows to prove a bulk-boundary correspondence for such systems [84] (talk by Stoiber). For graphene this allows to prove how the density of surface states of a half-space graphene sheet depends on the angle of the boundary and is dictated by weak (non-integer valued) bulk invariants. Another contribution showed that a fine tuning of the interaction and mass terms in a periodic system allows to construct interacting Weyl semimetals [35] (talk by Porta, which could also have been mentioned in the next section).

4 Recent Results on Interacting Topological Systems

After years of effort, there is by now a robust mathematical tool set to prove stability of gapped topological phases, both for quantum spin systems as well as fermionic systems [43, 62]. In concrete situations, one of the inputs is the proof of a bulk gap in the thermodynamic limit. Apart from the standard one-dimensional AKLT model, this has recently been achieved for a class of AKLT-like models in dimension two [1] (talk by Young). This is also connected to the stability of superselection sectors which has been shown recently for a class of (two-dimensional) dynamical toric code models, together with the invariance of the anyon fusion and statistics [22] (talk by Nachtergaele). A fruitful new direction to generate interesting topological models is supplied by the quantum deformations or quantum groups [76] (talk by Quella). Other contributions analyzed symmetry stabilized $\mathbb{Z}_2$-indices for interacting fermionic systems, albeit in dimension one where many results on quantum spin chains [26, 65] can be transposed using the Jordan-Wigner transformation [69, 70, 20] (talks by Ogata and Bourne).

Recently the concept of twisted bonds has been used effectively to define local topological indices in interacting systems (talk by Hatsugai). A similar approach made it possible (based on many prior works like [43, 62]) to define and prove many body index theorems for the Hall conductance of finite volume interacting fermionic systems with a good control on the error terms [6, 5, 66, 7] (talks by Avron, Bachmann and Bols).

One of the big open problems of the field of interacting fermion systems remains a robust argument as to why systems of interacting electrons in two dimensions and strong magnetic fields are so well described by Laughlin states. This is, of course, also linked to the fractional values of the Hall conductance. Several theoretical approaches are followed. A non-commutative geometry approach has been suggested and numerically supported [41] (talk by Haldane). A more geometric approach considers the motion of particles in a magnetic field on the maximal abelian cover of a compact Riemann surface [64] (talk by Matthai). Another geometric approach studies the adiabatic curvature and Quillen metric [56].

Topological order falls outside the bulk-boundary paradigm and is defined as the manifestation of a spectral degeneracy whenever a model is formulated over a surface of higher genus [91]. As a direct result, models with topological order have localized low energy excitations, called anyons, with non-trivial self-statistics [54]. The natural framework to describe and analyze these models seems to be tensor categories. The fundamental data for a topological order consists in the set of anyon type, their fusion rules and coefficients, S-matrix, braiding matrices and quantum dimensions (these are not all independent). Anyon braid matrices can be derived in microscopic models, but generating the fusion coefficients is a much more difficult task (talk by Levin). One of the fundamental applications proposed for the topological order is error correction in quantum computation [90] (talk by Mong). Spontaneous symmetry breaking from anyon condensation is connected to a short exact sequence whose splittings correspond to G-equivariant algebra structures. The non-splitting of this sequence forces spontaneous symmetry breaking under condensation of anyons, while inequivalent splittings of the sequence correspond to different symmetry enriched topological orders resulting from the anyon-condensation transition (talk by Lu).

5 Outcome of the Meeting

We plan to edit a special volume in the Journal of Geometry and Physics on the topic of the workshop. Many of the participants already have agreed to contribute to this. Let us add that a considerable number of the participants were scientist working in mathematical physics in Mexico and thus we hope that the meeting also will influence the scientific orientation of the community in Mexico.

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