# Mad families and the modal logic of $\mathbb{N}^*$

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August 7, 2019

Arch. Math. Logic (2009) 48:231–242
DOI 10.1007/s00153-009-0123-9

Mathematical Logic

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or Rather of $\beta\mathbb{N} \setminus \mathbb{N}$

is S4 if $\mathfrak{a} = \mathfrak{c}$ or if ...

We recall that **S4** is the least set of formulas containing the Boolean tautologies, the axioms:

$\Box(p \to q) \to (\Box p \to \Box q)$
$\Box p \to p$
$\Box\Box p \to \Box p$

and closed under Modus Ponens $(\varphi, \varphi \to \psi/\psi)$ and Necessitation $(\varphi/\Box\varphi)$. Rela-

McKinsey and Tarski defined a valuation $\nu$ of formulas of $\mathcal{L}$ into $\langle W, \tau\rangle$ by putting

- $\nu(p) \subseteq W$,
- $\nu(\neg\varphi) = W - \nu(\varphi)$,
- $\nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi)$,
- $\nu(\varphi \wedge \psi) = \nu(\varphi) \cap \nu(\psi)$,
- $\nu(\varphi \to \psi) = (W - \nu(\varphi)) \cup \nu(\psi)$,
- $\nu(\Box\varphi) = Int(\nu(\varphi))$,
- $\nu(\Diamond\varphi) = \overline{\nu(\varphi)}$.

In definitions and arguments in this paper, we will often economize, and leave out the clauses for disjunction, implication and modal diamond, as these are automatic from the others. Now, call a triple $M = \langle W, \tau, \nu\rangle$ a *topological model*. A formula $\varphi$ is said to be *true* in such a model $M$ if $\nu(\varphi) = W$, and we say that $\varphi$ is *topologically valid* if it is true in every topological model. Referring to the second axiomatization of **S4**, which highlights the interior operator, one easily sees its soundness:

If **S4** $\vdash \varphi$, then $\varphi$ is topologically valid.

the answer is YES (without assuming $\mathfrak{a} = \mathfrak{c}$) if the answer to this is YES

**Question**

Does $\mathbb{N}^*$ map onto every finite topological space by an open continuous map (with crowded fibers)

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[BH] proved that it suffices to work with finite $T_0$-spaces providing we have crowded fibers.

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**the $T_0$ Alexandroff topologies**

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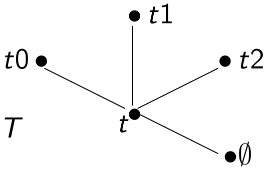
related to Veksler Problem: Can $\mathbb{N}^*$ have maximal nwd sets?

# branching is also hard

Consider $T$ 

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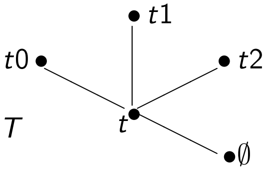


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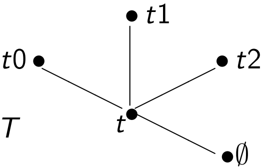


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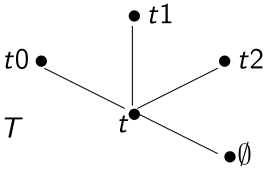


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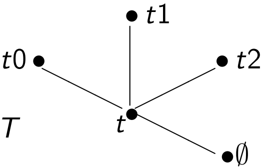


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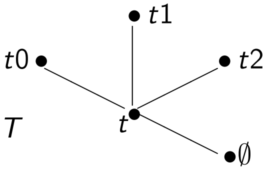


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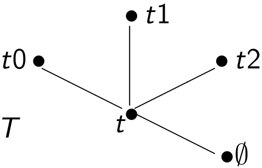


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Beszhanishvili-Harding used $\mathfrak{a} = \mathfrak{c}$, i.e. madf's

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**Simon: tfae**

1. There is no maximal nwd subset of $\mathbb{N}^*$
2. every madf has a completely separable madf refinement

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every infinite completely separable adf $\mathcal{A}$ is $+$-partitionable and $+$-refinable because $\mathfrak{c} = |\{a \subset^* X : a \in \mathcal{A}\}|$ for $X \in \mathcal{A}^+$

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2. Can $\mathcal{A}_1$ also be $+$-partitionable?

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Then, for all $X \in \mathcal{A}_0^+$, there is a $\alpha$ such that $X$ mod finite contains a member $b$ of $\mathcal{X}_\alpha$, and so,

## Proposition

If $\mathfrak{a} = \aleph_1$ (or $\mathfrak{a} = \mathfrak{h} = cof([\mathfrak{h}]^{\aleph_0})$), then $\mathcal{A}_1 \prec^+ \mathcal{A}_0$ exists.

but unlikely that $|\mathcal{A}_1| = \mathfrak{a}$ so no continuing

## Proof.

If $\mathcal{A}_0 = \{a_\alpha : \alpha \in \omega_1\}$, then for each $\alpha$ choose an almost disjoint refinement $\mathcal{X}_\alpha$ for $(\{a_\beta : \beta < \alpha\})^+$

so that $\mathcal{X}_\alpha \cup \{a_\beta : \beta < \alpha\}$ is an adf

(using Balcar-Simon tree $\pi$-base trick).

Choose a madf $\mathcal{A}_1(a_\alpha)$ on $a_\alpha$ so that, for each $\beta < \alpha$, each member of $\mathcal{X}_\beta \restriction a_\alpha$ contains infinitely many members of $\mathcal{A}_1(a_\alpha)$.

Then, for all $X \in \mathcal{A}_0^+$, there is a $\alpha$ such that $X$ mod finite contains a member $b$ of $\mathcal{X}_\alpha$, and so, there is a $\gamma \geqslant \alpha$ such that $b \cap a_\gamma$ is infinite and contains infinitely many members of $\mathcal{A}_1(a_\gamma)$. □

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*If there is a completely separable madf,*
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*hence $\mathbb{N}^*$ will map onto every $m^{\leqslant n}$ by an open continuous map.*

## Question

For which $n, m$ does such a family $\{\mathcal{A}_t : t \in m^{\leqslant n}\}$ exist?
Are there natural ZFC constructions?
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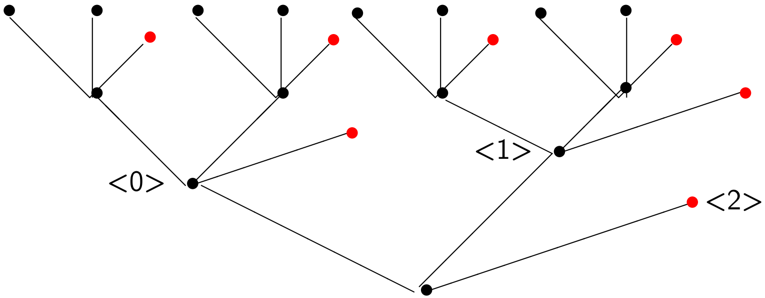
## Definition

Let $T_{n,m} = m^{\leqslant n} \cup \{t^\frown m : t \in m^{<n}\} \subset (m+1)^{\leqslant n}$

i.e. the subtree of $(m+1)^{\leqslant n}$ such that having $m$ in the range makes it a maximal node.

a picture of $T_{3,2}$



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## Proof.

Same construction except that, for $t \in m^{<n}$ and $a \in \mathcal{A}_t$, $\bigcup_{i<m} \mathcal{A}_{t^\frown i}(a)$ is a completely separable adf but not mad □

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**Now for some finite topology!**

$T_{n,m}$ maps onto $m^{\leqslant n}$ by an open continuous map.
Solving the Modal Logic problem.