$(*)$ 24/7

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Set theory of the reals

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The main result is joint work with Ralf Schindler.

Forcing axioms

Given a class $\mathcal{K}$ of forcing notions and a cardinal κ, $\mathsf{FA}_\kappa(\mathcal{K})$ is the following statement:

For every $\mathbb{P} \in \mathcal{K}$ and every collection $\{D_i : i < \kappa\}$ of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for each $i < \kappa$.

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Classical examples:

- MA_{ω_1} is $\mathsf{FA}_{\omega_1}(\{\mathbb{P} : \mathbb{P} \text{ ccc}\})$.
- PFA is $\mathsf{FA}_{\omega_1}(\{\mathbb{P} : \mathbb{P} \text{ proper}\})$.
- MM (Martin's Maximum) is $\mathsf{FA}_{\omega_1}(\{\mathbb{P} : \mathbb{P} \text{ semiproper}\})$ (equivalently, $\mathsf{FA}_{\omega_1}(\{\mathbb{P} : \mathbb{P} \text{ preserves stationary subsets of } \omega_1\})$).

Theorem (Foreman-Magidor-Shelah, 1984)

(1) *MM is a maximal forcing axiom: If $\mathbb{P}$ does not preserve stationary subsets of ω_1, then $FA_{\omega_1}(\{\mathbb{P}\})$ fails.*

(2) *MM, and in fact MM^{++}, can be forced assuming the existence of a supercompact cardinal.*

MM^{++} is the following strong form of MM: For every $\mathbb{P}$ preserving stationary subsets of ω_1, every $\{D_i : i < \omega_1\}$ consisting of dense subsets of $\mathbb{P}$ and every $\{\tau_i : i < \omega_1\}$ consisting of $\mathbb{P}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathbb{P}$ such that

- $G \cap D_i \neq \emptyset$ for each $i < \omega_1$, and
- $\{\nu < \omega_1 : (\exists p \in G)\, p \Vdash_{\mathbb{P}} \nu \in \tau_i\}$ is a stationary subset of ω_1 for each $i < \omega_1$.

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MM^{++} has many consequences for $H(\omega_2)$:

- $\mathfrak{p} = 2^{\aleph_0} = \aleph_2$ and there is a simply boldface definable (over $H(\omega_2)$) well-order of $H(\omega_2)$ of length ω_2 (MA_{ω_1} [folklore?] and PFA [Todorčević, Veličković, and Moore], resp.)
- All $\aleph_1$–dense sets of reals are order–isomorphic. (PFA [Baumgartner])
- There is a 5-element basis for the uncountable linear orders. (PFA [Moore])
- $\delta^1_2 = \omega_2$ (MM [Woodin])
- ...

Empirical fact: MM^{++} seems to provide a complete theory for $H(\omega_2)$ modulo forcing (on the other hand, MM, or even $\mathsf{MM}^{+\omega}$, does not [Larson]).

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In the 1990's, Woodin defined and studied the following axiom.[1]

$(*)$: AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$–extension of $L(\mathbb{R})$.

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Given $\eta \leq \omega_1$, a sequence $(\langle (M_\alpha, I_\alpha), G_\alpha, j_{\alpha,\beta}\rangle : \alpha < \beta \leq \eta)$ is a *generic iteration (of (M_0, I_0))* iff

- M_0 is a countable transitive model of ZFC^* (enough of ZFC).
- $I_0 \in M_0$ is, in M_0, a normal ideal on $\omega_1^{M_0}$.
- $j_{\alpha,\beta}$, for $\alpha < \beta \leq \eta$, is a commuting system of elementary embeddings

$$j_{\alpha,\beta} : (M_\alpha; \in, I_\alpha) \longrightarrow (M_\beta, \in, I_\beta)$$

- For each $\alpha < \eta$, G_α is a $\mathcal{P}(\omega_1)^{M_\alpha}/I_\alpha$–generic filter over M_α,

$$j_{\alpha,\alpha+1} : M_\alpha \longrightarrow \mathrm{Ult}(M_\alpha, G_\alpha)$$

  is the corresponding elementary embedding, and $(M_{\alpha+1}, I_{\alpha+1}) = (\mathrm{Ult}(M_\alpha, G_\alpha), j_{\alpha,\alpha+1}(I_\alpha))$.
- If $\beta \leq \eta$ is a limit ordinal, (M_β, I_β) and $j_{\alpha,\beta}$ (for $\alpha < \beta$) is the direct limit of $(\langle (M_\alpha, I_\alpha), G_\alpha, j_{\alpha,\alpha'}\rangle : \alpha < \alpha' < \beta)$.

A pair (M, I) is *iterable* if the models in every generic iteration of (M, I) are well–founded.

$\mathbb{P}_{\max}$ is the following forcing:

Conditions in $\mathbb{P}_{\max}$ are triples (M, I, a), where

(1) (M, I) is an iterable pair.

(2) $M \models \mathrm{MA}_{\omega_1}$

(3) $a \in \mathcal{P}(\omega_1)^M$ and $M \models \omega_1 = \omega_1^{L[a]}$.

Extension relation: $(M^1, I^1, a^1) \leq_{\mathbb{P}_{\max}} (M^0, I^0, a^0)$ iff $(M^0, I^0, a^0) \in M_1$ and, in M^1, there is a generic iteration $\mathcal{I} = (\langle (M_\alpha, I_\alpha), G_\alpha, j_{\alpha,\beta} \rangle : \alpha < \beta \leq \eta)$ of (M^0, I^0) for $\eta = \omega_1^{M^1}$ such that

(a) $j_{0,\eta}(a^0) = a^1$

(b) $\mathcal{I}$ is *correct* in (M^1, I^1), in the sense that $j_{0,\eta}(I^0) \subseteq I^1$ and every I_η–positive subset of $\omega_1^{M_\eta}$ $(= \omega_1^{M^1})$ in M_η is I^1–positive.

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Some properties of $\mathbb{P}_{max}$ under $\mathsf{AD}^{L(\mathbb{R})}$:

- $\mathbb{P}_{max}$ is weakly homogeneous (for all $p_0, p_1 \in \mathbb{P}_{max}$ there are $p_0' \leq_{\mathbb{P}_{max}} p_0$ and $p_1' \leq_{\mathbb{P}_{max}} p_1$ such that $\mathbb{P}_{max} \restriction p_0' \cong \mathbb{P}_{max} \restriction p_1'$).
- $\mathbb{P}_{max}$ is σ–closed (in particular it does not add new reals).

- If G is $\mathbb{P}_{max}$–generic over $L(\mathbb{R})$, then $L(\mathbb{R})[G] \models$ ZFC, and if

  $$A_G = \bigcup\{b : (N, J, b) \in G\},$$

  G can be computed in $L(\mathbb{R})[A_G]$ as the set Γ_{A_G} of $(M, I, b) \in \mathbb{P}_{max}$ such that there is a correct iteration (relative to $(H(\omega_2), \mathrm{NS}_{\omega_1})$) sending b to A_G.

  If fact, for any $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$, Γ_A can be computed in $L(\mathbb{R})[A]$, Γ_A is a $\mathbb{P}_{max}$–generic filter over $L(\mathbb{R})$, and

  $$L(\mathbb{R})[\Gamma_A] = L(\mathbb{R})[G]$$

  In particular, $L(\mathbb{R})[G] \models V = L(\mathcal{P}(\omega_1))$, and so $L(\mathbb{R})[G] \models (*)$ if $L(\mathbb{R}) \models$ AD and G is $\mathbb{P}_{max}$–generic over $L(\mathbb{R})$.

- (Π_2 maximality) Assuming enough large cardinals (e.g. a proper class of Woodin cardinal). If G is $\mathbb{P}_{max}$–generic over $L(\mathbb{R})$, $\mathcal{Q}$ is a set–forcing in V, H is $\mathcal{Q}$–generic over V, and σ is a Π_2 sentence such that

$$(H(\omega_2), \in, \mathrm{NS}_{\omega_1})^{V[H]} \models \sigma,$$

then

$$(H(\omega_2), \in, \mathrm{NS}_{\omega_1})^{L(\mathbb{R})[G]} \models \sigma$$

- (Completeness modulo set-forcing) Assuming enough large cardinals (e.g. a proper class of Woodin cardinal). Let $\mathcal{Q}_0$ and $\mathcal{Q}_1$ be set-forcings in V, let H_0 be $\mathcal{Q}_0$–generic over V and H_1 be $\mathcal{Q}_1$–generic over V, and let G_0 be $\mathbb{P}_{max}^{L(\mathbb{R}^{V[H_0]})}$–generic over $L(\mathbb{R}^{V[H_0]})$ and G_1 be $\mathbb{P}_{max}^{L(\mathbb{R}^{V[H_1]})}$–generic over $L(\mathbb{R}^{V[H_1]})$. Then

$$\mathrm{Th}(L(\mathbb{R}^{V[H_0]})[G_0]) = \mathrm{Th}(L(\mathbb{R}^{V[H_1]})[G_1])$$

Proof of the completeness result: Let σ be any sentence and suppose

$$L(\mathbb{R}^{V[H_0]})[G_0] \models \sigma$$

By weak homogeneity of $\mathbb{P}_{max}$,

$$L(\mathbb{R}^{V[H_0]}) \models \text{``} \Vdash_{\mathbb{P}_{max}} \sigma \text{''}$$

But the theory of $L(\mathbb{R})$ is invariant under forcing with our background large cardinals. Hence,

$$L(\mathbb{R}^{V[H_1]}) \models \text{``} \Vdash_{\mathbb{P}_{max}} \sigma \text{''}$$

and therefore

$$L(\mathbb{R}^{V[H_1]})[G_1] \models \sigma$$

$\square$

Some consequences of $(*)$:

- $\mathfrak{p} = 2^{\aleph_0} = \aleph_2$ and there is a simply boldface definable (over $H(\omega_2)$) well–order of $H(\omega_2)$ of length ω_2.
- All $\aleph_1$–dense sets of reals are order–isomorphic.
- There is a 5-element basis for the uncountable linear orders.

- $\delta^1_2 = \omega_2$
- $\ldots$

So $(*)$ and forcing axioms in the region of MM seem to be closely related. However, $\mathrm{MM}^{+\omega}$ does **not** imply $(*)$: $\mathrm{MM}^{+\omega}$ is consistent with a lightface definable well–order, over $H(\omega_2)$, of $H(\omega_2)$ [Larson], which cannot exist under $(*)$. Otherwise by weak homogeneity of $\mathbb{P}_{max}$ there would be a well–order of $\mathbb{R}$ in $L(\mathbb{R})$, contradicting $\mathrm{AD}^{L(\mathbb{R})}$.

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More consequences of $(*)$:

- For every $X \subseteq \omega_1$ such that $X \notin L[x]$ for any $x \in \mathbb{R}$ there is a real r and a $\mathrm{Coll}(\omega, <\omega_1)$–generic filter H over $L[r]$ such that $L[r][X] = L[r][H]$.

- For every $X \subseteq \omega_1$ there is $Y \subseteq \omega_1$ such that $X \in L[Y]$ and such that for every $Z \subseteq \omega_1$, if $Z \cap \alpha \in L[Y]$ for all $\alpha < \omega_1$, then $Z \in L[Y]$.

$(*)$ is NICE

To summarize:

(1) (Π_2**–maximality**) $(*)$ + large cardinals implies that $(H(\omega_2); \in, \mathrm{NS}_{\omega_1})$ satisfies all forcible Π_2 sentences over $(H(\omega_2); \in, \mathrm{NS}_{\omega_1})$.

(2) (**Completeness**) $(*)$ + large cardinals provides a complete theory for $L(\mathcal{P}(\omega_1))$, modulo set-forcing.

(3) (**Minimality**) $(*)$ implies that $L(\mathcal{P}_{\omega_1})$ is a "canonical" model; in fact, of the form $L(\mathbb{R})[H]$ for any $r \in \mathbb{R}$ and any $\mathrm{Coll}(\omega, {<}\omega_1)$–generic H over $L[r]$.

But in order for $(*)$ to be strongly NICE, it would have to be compatible with all possible large cardinals.

Question (Woodin): Is $(*)$ compatible with all possible large cardinals? Does in fact $(*)$ follow from MM^{++}?

The main result

Theorem (A–Schindler)

MM^{++} implies $(*)$.

A related result

Theorem (Todorčević)

Assume all sets of reals in $L(\mathbb{R})$ *are universally Baire. If* $\mathcal{U}$ *is a Ramsey ultrafilter, then* $\mathcal{U}$ *is* $\mathcal{P}(\omega)/Fin$*–generic over* $L(\mathbb{R})$*.*

In the rest of the talk, I will sketch the proof of our theorem. As we will see, the main idea is to use “iterated $\mathcal{L}$–forcing” with side conditions.

MM^{++} implies AD$^{L(\mathbb{R})}$ (PFA suffices), so we only need to show that $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$–extension on $L(\mathbb{R})$.

It is well-known that if NS_{ω_1} is saturated, MA_{ω_1} holds, $\mathcal{P}(\omega_1)^\sharp$ exists, and $A \subseteq \omega_1$ is such that $\omega_1^{L[A]} = \omega_1$, then Γ_A is a filter on $\mathbb{P}_{\max}$ and $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[\Gamma_A]$.

Since MM^{++} implies the hypotheses (in fact MM does), it suffices to assume MM^{++} and prove that Γ_A is in fact $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$.

So let $D \in L(\mathbb{R})$ be a dense subset of $\mathbb{P}_{max}$. We will prove that $\Gamma_A \cap D \neq \emptyset$.

MM^{++} implies that every set of reals in $L(\mathbb{R})$ is universally Baire and the class of sets of reals in $L(\mathbb{R})$ is productive, so we may fix a tree T on $\omega \times 2^{\aleph_2}$ such that $p[T]$ is (a set of reals coding the members of) D and such that

$\Vdash_{\mathrm{Coll}(\omega,\omega_2)}$ "$p[T]$ codes the members of a dense subset of $\mathbb{P}_{max}$"

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$\Vdash_{\mathrm{Coll}(\omega,\omega_2)}$ "$p[T]$ codes the members of a dense subset of $\mathbb{P}_{max}$"

It suffices to show that there is a forcing $\mathcal{Q}$ preserving stationary subsets of ω_1 and forcing that there is a branch $[x, b]$ through T such that x codes a member of Γ_A.

Let $\kappa = (2^{\aleph_2})^+$. Let d be $\text{Coll}(\kappa, \kappa)$–generic over V. In $V[d]$ there is a club $D \subseteq \kappa$ of ordinals above ω_2 and a 'diamond sequence'

$$(\langle Q_\lambda, B_\lambda \rangle \,:\, \lambda \in C)$$

such that $(Q_\lambda \,:\, \lambda \in C)$ is a strictly $\subseteq$–increasing and $\subseteq$–continuous seq. of transitive elem. submodels of $H(\kappa)^{V[d]} = H(\kappa)^V$ and $B_\lambda \subseteq Q_\lambda$ for all $\lambda \in C$.

Enough to show there is in $V[d]$ a forcing $\mathcal{P}$ preserving stationary subsets of ω_1 and forcing that there is a branch $[x, b]$ through T such that x codes a member of Γ_A. (Hence I'll be writting V for $V[d]$.)

$\mathcal{P}$ will be $\mathcal{P}_\kappa$, where

$$(\mathcal{P}_\lambda \,:\, \lambda \in C \cup \{\kappa\})$$

is the sequence of forcings defined by letting $\mathcal{P}_\lambda$ be the set, ordered under $\supseteq$, of finite sets p of sentences, in a suitable fixed language, such that $\mathrm{Coll}(\omega, \lambda)$ forces that there is a λ*–certificate for* p.

λ–certificates

A λ*–pre-certificate (relative to* $(H(\omega_2)^V; \in, \mathrm{NS}^V_{\omega_1}, A)$ *and* T*)* is a complete set Σ of sentences, in a suitable fixed language, describing finitary information about the following objects.

(1) $\mathcal{M}_0, \mathcal{N}_0 \in \mathbb{P}_{\max}$

(2) $x = \langle k_n : n < \omega \rangle$, a real coding N_0, and $\langle (k_n, \alpha_n) : n < \omega \rangle$, a branch through T.

(3) $\langle \mathcal{M}_i, \pi_{i,j} : i \leq j \leq \omega_1^{N_0} \rangle \in N_0$, a generic iteration of $\mathcal{M}_0$ witnessing $\mathcal{N}_0 \leq_{\mathbb{P}_{\max}} \mathcal{M}_0$.

(4) $\langle \mathcal{N}_i, \sigma_{i,j} : i \leq j \leq \omega_1 \rangle$, a generic iteration of $\mathcal{N}_0$ such that if

$$\mathcal{N}_{\omega_1} = (N_{\omega_1}; \in, I^*, A^*),$$

then $A^* = A$.

(5) $\langle \mathcal{M}_i, \pi_{i,j} : i \leq j \leq \omega_1 \rangle = \sigma_{0,\omega_1}(\langle \mathcal{M}_i, \pi_{i,j} : i \leq j \leq \omega_1^{\mathcal{N}_0} \rangle)$ and

$$\mathcal{M}_{\omega_1} = (H(\omega_2)^V; \in, \mathrm{NS}^V_{\omega_1}, A)$$

(6) $K \subset \omega_1$, and for all $\delta \in K$,

(a) $\lambda_\delta \in C \cap \lambda$, and if $\gamma < \delta$ is in K, then $\lambda_\gamma < \lambda_\delta$ and $X_\gamma \cup \{\lambda_\gamma\} \subset X_\delta$,

(b) $X_\delta \prec (Q_{\lambda_\delta}; \in, \mathcal{P}_{\lambda_\delta}, B_{\lambda_\delta})$, and

(c) $X_\delta \cap \omega_1 = \delta$

A λ–pre-certificate Σ is a λ–*certificate* if, in addition:

($\triangle$) For every $\delta \in K$,

$$[\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset$$

for every dense $E \subseteq \mathcal{P}_\delta$ definable over the structure

$$(Q_{\lambda_\delta}; \in, \mathcal{P}_{\lambda_\delta}, B_{\lambda_\delta})$$

from parameters in X_δ.

A condition in $\mathcal{P}_\lambda$ is a finite set p of sentences such that

$\Vdash_{\mathrm{Coll}(\omega,\lambda)}$ "There is a λ–certificate Σ such that $p \in [\Sigma]^{<\omega}$"

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• $(\mathcal{P}_\lambda : \lambda \in C \cup \{\kappa\})$ is an $\subseteq$–increasing and $\subseteq$–continuous seq. of forcings and $\mathcal{P}_\kappa \subseteq H(\kappa)^V$.

• For every $\lambda \in C$, $\mathcal{P}_\lambda \neq \emptyset$: Let g be $\mathrm{Coll}(\omega, \omega_2)$-generic over V. Then

$$\mathcal{M}_0 = (H(\omega_2)^V; \in, \mathrm{NS}^V_{\omega_1})$$

is a $\mathbb{P}_{\max}$–condition. Since $p[T]$ is a dense subset of $\mathbb{P}_{\max}$ in $V[g]$, there is in $V[g]$ a branch $\langle (k_n)_{n<\omega}, (\alpha_n)_{n<\omega} \rangle$ of T with $(k_n)_{n<\omega}$ coding $\mathcal{N}_0 \in \mathbb{P}_{\max}$, together with a correct iteration $\mathcal{I}_0 = \langle \mathcal{M}_i, \pi_{i,j} : i \leq j \leq \omega_1^{\mathcal{N}_0} \rangle \in N_0$ of $\mathcal{M}_0$ witnessing $\mathcal{N}_0 \leq_{\mathbb{P}_{\max}} \mathcal{M}_0$.

In $V[g]$, let $(\mathcal{N}_i, \sigma_{i,j} : i < j \leq \omega_1)$ be a generic iteration of $\mathcal{N}_0$. Let $\mathcal{I} = (\mathcal{M}_i, \pi_{i,j} : i < j \leq \omega_1) = \sigma_{0,\omega_1}(\mathcal{I}_0)$.

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$\mathcal{I}$ lifts to a generic iteration $(\mathcal{M}_i^+, \pi_{i,j}^+ : i < j \leq \omega_1)$ of V. Let $M = M_{\omega_1}^+$ and $\pi = \pi_{0,\omega_1}^+$. The theory of

$$\langle M_i, \pi_{i,j}, N_i, \sigma_{i,j} : i < j \leq \omega_1 \rangle, \langle (k_n)_{n<\omega}, (\pi(\alpha_n))_{n<\omega} \rangle, \langle \rangle$$

is a λ–certificate for $\emptyset$, relative to $\pi((H(\omega_2)^V; \in, \mathrm{NS}_{\omega_1}^V, A))$ and $\pi(T)$, in some outer model. But then there is a λ–certificate for $\emptyset$, relative to $\pi((H(\omega_2)^V; \in, \mathrm{NS}_{\omega_1}^V, A))$ and $\pi(T)$, in $M^{\mathrm{Coll}(\omega, \pi(\lambda))}$ by Σ_1^1–absoluteness, and the same is true in $V^{\mathrm{Coll}(\omega, \lambda)}$, relative to $(H(\omega_2)^V; \in, \mathrm{NS}_{\omega_1}^V, A)$ and T, by elementarity of π. $\square$

• Standard density argument show that if G is $\mathcal{P}$-generic over V and

$$\langle \mathcal{M}_i, \pi_{i,j}, \mathcal{N}_i, \sigma_{i,j} \colon i < j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle$$

is the term model given by $\Sigma := \bigcup G$, then

$$\mathcal{I} = \langle \mathcal{N}_i, \sigma_{i,j} \ : \ i < j \leq \omega_1 \rangle$$

is a generic iteration such that

- $H(\omega_2)^V \subseteq N_{\omega_1}$,
- $(\mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1})^V \subseteq \mathcal{P}(\omega_1)^{\mathcal{N}_{\omega_1}} \setminus I_{\mathcal{N}_{\omega_1}}$,
- $A_{\mathcal{N}_{\omega_1}} = A$, and
- $\mathcal{N}_0$ is coded by a real in $p[T]$.

Crucial lemma

Lemma

If $S \in \mathcal{P}(\omega_1)^{\mathcal{N}_{\omega_1}} \setminus I_{\mathcal{N}_{\omega_1}}$, then S is stationary in $V[G]$.

[This immediately implies that $\mathcal{I}$ is correct in $V[G]$ and that $\mathcal{P}$ preserves stationary subsets of V.]

Proof sketch of Lemma: Let $\dot{C}$ be a $\mathcal{P}$–name for a club, $\dot{S}$ a $\mathcal{P}$–name for set in $\mathcal{P}(\omega_1)^{\mathcal{N}_{\omega_1}} \setminus I_{\mathcal{N}_{\omega_1}}$, and $p \in \mathcal{P}$. Let $\lambda \in C$ such that B_λ codes $\dot{C} \cap (\mathcal{P}_\lambda \times \omega_1)$ and

$$(Q_\lambda; \in, \mathcal{P}_\lambda, \dot{C} \cap \mathcal{P}_\lambda) \prec (H(\kappa)^V; \in, \mathcal{P}, \dot{C})$$

Working in collapse W of V with $\omega_1^V < \omega_1^W$, find a $\mathcal{P}_\lambda$–generic filter G over V with $p \in G$. Let

$$\langle \mathcal{M}_i, \pi_{i,j} : i < j < \omega_1^V \rangle, \langle \mathcal{N}_i, \sigma_{i,j} : i < j < \omega_1^V \rangle, \ldots$$

be the corresponding objects given by G.

We may extend

$$\langle \mathcal{N}_i, \sigma_{i,j} \ : \ i < j < \omega_1^V \rangle$$

to

$$\langle \mathcal{N}_i, \sigma_{i,j} \ : \ i < j < \omega_1^W \rangle$$

such that $\delta = \omega_1^V \in \sigma_{\omega_1^V, \omega_1^W}(\dot{S})$.

By an elementarity argument as in the proof that $\mathcal{P}_\lambda \neq \emptyset$, there is, in V, some $q^* \leq_{\mathbb{P}_{\max}} q$ for which there is some

$$\delta \in K^{q^*}$$

which q^* enforces to be in $\dot{S}$ and such that

$$\lambda_\delta = \lambda$$

(For example existence of X_δ is witnessed by $\pi``Q_\lambda$.)

But since

$$(Q_\lambda; \in, \mathcal{P}_\lambda, \dot{C} \cap \mathcal{P}_\lambda) \prec (H(\kappa)^V; \in, \mathcal{P}, \dot{C}),$$

by a density argument q^* forces that δ is a limit point of $\dot{C}$, and hence in $\dot{C}$. Clause $(\triangle)$ is used crucially for this:

Given any $q' \leq_{\mathcal{P}} q^*$ and $\xi < \delta$, any κ–certificate Σ for q' will contain $p \in X_\delta$ forcing some ordinal $\xi' > \xi$ in $\dot{C}$ (thanks to $(\triangle)$), since

$$\{r \in \mathcal{P}_\lambda \,:\, (\exists \xi' > \xi) r \Vdash_{\mathcal{P}_\lambda} \xi' \in \dot{C}\}$$

is a dense set definable over

$$(Q_\lambda; \in, \mathcal{P}_\lambda, B_\lambda)$$

from $\xi \in X_\delta$). Of course $\xi' < \delta$ since $p \in X_\delta$ and $X_\delta \cap \omega_1 = \delta$. But then $p \cup q'$ is a common extension of p and q' in $\mathcal{P}$. $\square$

Corollary

MM^{++} implies the following.

- *For every $X \subseteq \omega_1$ such that $X \notin L[x]$ for any $x \in \mathbb{R}$ there is a real r and a $\mathrm{Coll}(\omega, {<}\omega_1)$–generic filter H over $L[r]$ such that $L[r][X] = L[r][H]$.*

- *For every $X \subseteq \omega_1$ there is $Y \subseteq \omega_1$ such that $X \in L[Y]$ and such that for every $Z \subseteq \omega_1$, if $Z \cap \alpha \in L[Y]$ for all $\alpha < \omega_1$, then $Z \in L[Y]$.*

Thank you!