Preservation theorems for finite support iterations

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Set Theory of the Reals

Oaxaca, México
August 5th, 2019

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$\mathcal{M}$: the ideal of first category subsets of $\mathbb{R}$.

$\mathcal{N}$: the ideal of Lebesgue measure zero subsets of $\mathbb{R}$.

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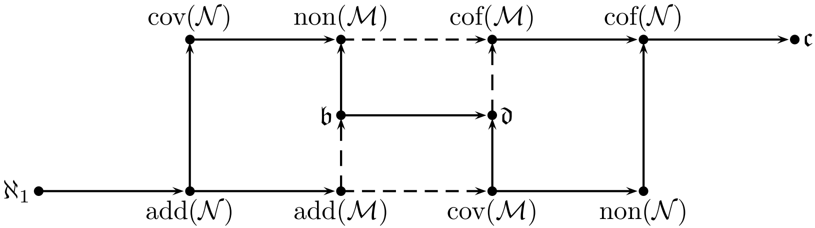
Consider

$$\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega \text{ and } \neg(\exists g \in \omega^\omega)(\forall f \in F)\, f \leq^* g\}$$
$$\mathfrak{d} = \min\{|D| : D \subseteq \omega^\omega \text{ and } (\forall f \in \omega^\omega)(\exists g \in D)\, f \leq^* g\}$$
$$\mathfrak{c} = 2^{\aleph_0}$$

Cichoń's diagram

Inequalities: **Bartoszyński, Fremlin, Miller, Rothberger, Truss**.
Completeness: **Bartoszyński, Judah, Miller, Shelah**.



Also $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

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Playground

Cichoń's diagram (just the left side).

General framework

Relational system

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6. $\mathbf{C}_\mathcal{I} := \langle X, \mathcal{I}, \in \rangle$, $\mathfrak{b}(\mathbf{C}_\mathcal{I}) = \mathrm{non}(\mathcal{I})$, $\mathfrak{d}(\mathbf{C}_\mathcal{I}) = \mathrm{cov}(\mathcal{I})$.

Dual and Tukey connections

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Example

For $h \in \omega^\omega$ let $\mathbf{Lc}(h) = \langle \omega^\omega, \left([\omega]^{<\aleph_0}\right)^\omega, \in^*_h \rangle$ where

$$x \in^*_h \varphi \text{ iff } (\forall i)\, |\varphi(i)| \leq h(i), \text{ and } (\exists m)(\forall i \geq m)\, x(i) \in \varphi(i).$$

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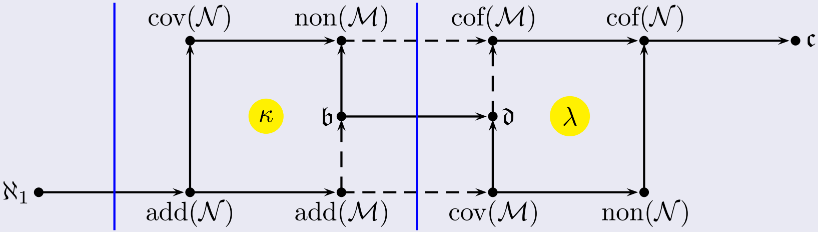
Bartoszyński (1984)

If $h \to \infty$ then $\mathcal{N} \cong_{\mathrm{T}} \mathbf{Lc}(h)$. Hence $\mathfrak{b}(\mathbf{Lc}(h)) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc}(h)) = \mathrm{cof}(\mathcal{N})$.

Example of 3 values

Theorem (Brendle 1991)

If $\kappa \geq \aleph_1$ is regular and $\lambda = \lambda^{<\kappa}$, then it is consistent with ZFC that



Localization forcing

Define the poset $\mathbb{LOC}$:

Conditions. $\varphi : \omega \to [\omega]^{<\aleph_0}$ such that $(\forall i)\, |\varphi(i)| \leq i$, and $(\exists m)(\forall i \geq m)\, |\varphi(i)| \leq m$.

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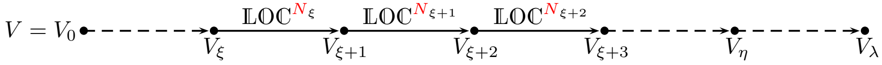
3. If $N \subseteq V$ is a transitive model (of ZFC) then $\mathbb{LOC}^N$ is still σ-linked.

Proof 1 (half)

Perform a FS iteration of length λ using

$\mathbb{LOC}^{N_\xi}$ for some transitive model N_ξ of size $<\kappa$.



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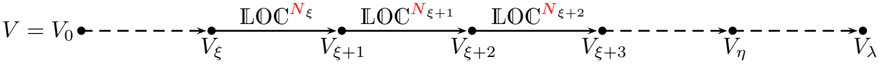

Book-keeping argument: Any $\mathbb{P}$-name $\dot{Z} \subseteq \omega^\omega$, $|\dot{Z}| < \kappa$ is contained in some N_ξ ($\xi < \lambda$) (OK because $\lambda^{<\kappa} = \lambda$).

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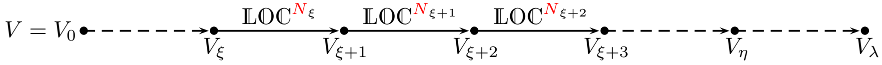

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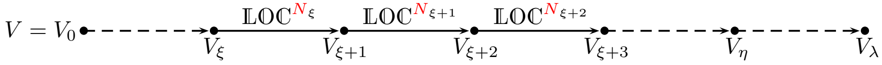

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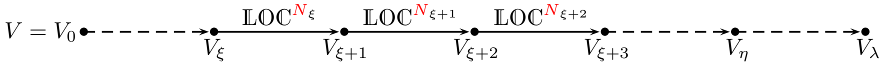

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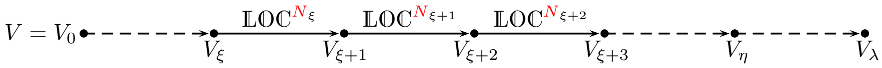

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Hence $\kappa \leq \mathrm{add}(\mathcal{N})$.

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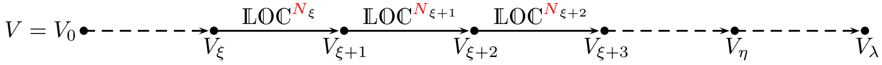
$\mathbb{LOC}^{N_\xi}$ for some transitive model N_ξ of size $<\kappa$.



Book-keeping argument: Any $\mathbb{P}$-name $\dot{Z} \subseteq \omega^\omega$, $|\dot{Z}| < \kappa$ is contained in some N_ξ $(\xi < \lambda)$ (OK because $\lambda^{<\kappa} = \lambda$).

In V_λ, for any $\xi < \lambda$ let φ_ξ be the slalom added by $\mathbb{LOC}^{N_\xi}$.

1. $(\forall x \in \omega^\omega \cap N_\xi)\, x \in^*_{\mathrm{id}} \varphi_\xi$.
2. For any $Z \subseteq \omega^\omega$ of size $<\kappa$, there is some $\xi < \lambda$ such that $(\forall x \in Z)\, x \in^*_{\mathrm{id}} \varphi_\xi$.

Hence $\kappa \leq \mathrm{add}(\mathcal{N})$. On the other hand, $\mathfrak{c} \leq \lambda$.

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θ-R-DOM

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If $\exists D \subseteq Y$ θ-**R**-DOM, then $\theta \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq |D|$.

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Also useful for θ-$\mathbf{R}^\perp$-DOM.

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Fix a directed order $S = \langle S, \leq \rangle$.

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Fact

($\exists$ S-$\mathbf{R}$-COB set) iff $\mathbf{R} \preceq_{\mathrm{T}} S$, and each implies $\mathrm{cp}(S) \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathrm{cf}(S)$.

Special dominating families 2 (cont.)

Notation

Say that $\mathbf{R} = \langle X, Y, R \rangle$ is Borel if X, Y are Polish spaces and R is Borel.

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$\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ means that there is some $\{\dot{y}_i : i \in S\} \subseteq Y$ s.t.

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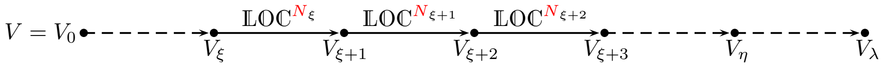

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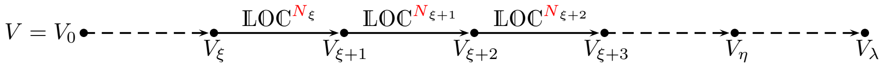

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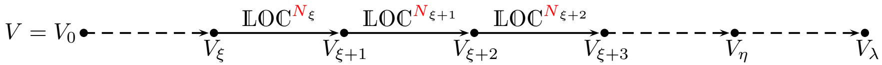

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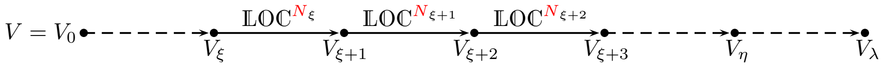

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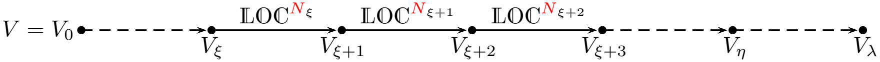

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3. $\mathrm{COB}_{\mathbf{Lc}(\mathrm{id})}(\mathbb{P}, S)$ holds. So $\mathbb{P}$ forces $\kappa \leq \mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N}) \leq \lambda$.

Cohen reals

There is an F_σ relation $R_4 \subseteq 2^\omega$ such that $\mathbf{R}_4 := \langle 2^\omega, 2^\omega, R_4 \rangle \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}}$. Hence $\mathfrak{b}(\mathbf{R}_4) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{R}) = \mathrm{cov}(\mathcal{M})$

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4. $\mathbb{P}$ forces $\mathfrak{b}(\mathbf{R}_4) \leq \mathrm{cf}(\theta) = \theta \leq \mathfrak{d}(\mathbf{R}_4)$.

Special unbounded families

Fix a Borel $\mathbf{R} = \langle X, Y, R \rangle$.

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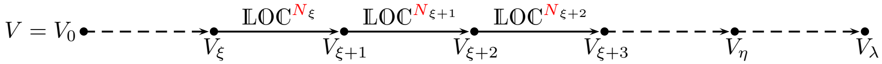
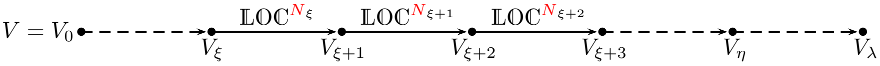


Main Claim

For any regular $\kappa \leq \theta \leq \lambda$, the θ-$\mathbf{R}_4$-LCU set of Cohen reals added by $\mathbb{P}_\theta$ is preserved in V_λ. I.e., $\mathrm{LCU}(\mathbb{P}, \theta)$ holds.

Proof 1 & 2 (cont.)



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For such θ, $\mathbf{R}_4^\perp \preceq_{\mathrm{T}} \theta$, so $\mathrm{non}(\mathcal{M}) \leq \theta \leq \mathrm{cov}(\mathcal{M})$.

Hence $\mathrm{non}(\mathcal{M}) \leq \kappa$ and $\lambda \leq \mathrm{cov}(\mathcal{M})$.

Preservation theory 1

Fix a Borel $\mathbf{R}$ and $\theta \geq \aleph_1$ regular.

Definition (Judah & Shelah 1990, and Brendle 1991)

A poset $\mathbb{P}$ is θ-$\mathbf{R}$-good if $(\forall \dot{y} \in Y)(\exists H \subseteq Y)$:

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Fact

If $\mathbb{P}$ is θ-cc and θ-$\mathbf{R}$-good then it preserves

(i) θ-$\mathbf{R}^{\perp}$-DOM sets,

(ii) S-$\mathbf{R}^{\perp}$-COB sets whenever $\mathrm{cp}(S) \geq \theta$,

(iii) L-$\mathbf{R}$-LCU sets whenever $\mathrm{cf}(L) \geq \theta$.

Preservation theory 1

Fix a Borel $\mathbf{R}$ and $\theta \geq \aleph_1$ regular.

Definition (Judah & Shelah 1990, and Brendle 1991)

A poset $\mathbb{P}$ is θ-$\mathbf{R}$-good if $(\forall \dot{y} \in Y)(\exists H \subseteq Y)$:

$$0 < |H| < \theta \text{ and } (\forall x \in X)\Big[\big((\forall z \in H)\,\neg(xRz)\big) \Rightarrow \Vdash_{\mathbb{P}} \neg(xR\dot{y})\Big]$$

Fact

If $\mathbb{P}$ is θ-cc and θ-$\mathbf{R}$-good then it preserves

(i) θ-$\mathbf{R}^{\perp}$-DOM sets,

(ii) S-$\mathbf{R}^{\perp}$-COB sets whenever $\mathrm{cp}(S) \geq \theta$,

(iii) L-$\mathbf{R}$-LCU sets whenever $\mathrm{cf}(L) \geq \theta$.

1. If $\mathbb{P}$ is θ-$\mathbf{R}$-good and $\theta \leq \theta'$ then $\mathbb{P}$ is θ'-$\mathbf{R}$-good.
2. If $\mathbb{P}$ is θ-$\mathbf{R}$-good and $\mathbb{P}_0 \lessdot \mathbb{P}$ then $\mathbb{P}_0$ is.

Preservation theory 1 (cont.)

Definition

Say that $\mathbf{R} = \langle X, Y, R\rangle$ is Polish if

1. X is perfect Polish, Y is Polish,
2. $R = \bigcup_{i<\omega} R_i$ where each R_i is closed,
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Fix a Polish $\mathbf{R}$ and an uncountable regular κ.

Theorem

Any FS iteration of κ-cc κ-$\mathbf{R}$-good posets is again κ-$\mathbf{R}$-good.

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Corollary

If $\mathbb{P}$ is a FS iteration of κ-cc κ-$\mathbf{R}$-good posets then $\mathrm{LCU}_{\mathbf{R}}(\mathbb{P}, \theta)$ holds for any regular $\kappa \leq \theta \leq$ length.

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For any Polish $\mathbf{R}$ and $\theta \geq \aleph_1$ regular, if $|\mathbb{P}| < \theta$ then $\mathbb{P}$ is θ-$\mathbf{R}$-good.

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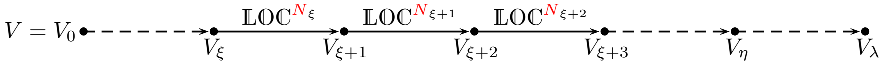
Since $\mathbf{R}_4$ is Polish,



is κ-$\mathbf{R}_4$-good. This proves the Main Claim before.

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   (Kamburelis 1989) Any subalgebra of random forcing is $\aleph_1$-$\mathbf{R}_1$-good.

Examples

Let $\mathbf{R}_{\mathrm{rp}} := \langle [\omega]^{\aleph_0}, [\omega]^{\aleph_0}, R_{\mathrm{rp}} \rangle$ where

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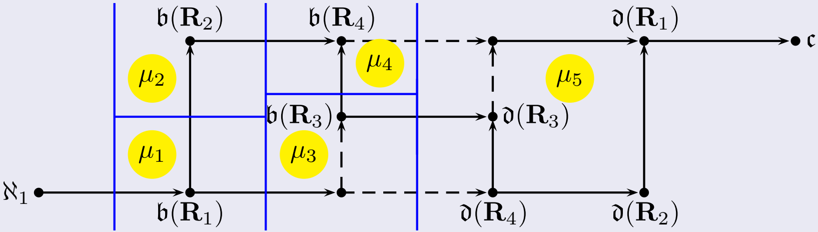
Dow & Shelah 2018

If F is a filter on ω generated by $<\theta$ many sets then $\mathbb{L}_F$ is θ-$\mathbf{R}_{\mathrm{rp}}$-good.

Left side (6 values)

Theorem (Goldstern & M. & Shelah 2016)

Let $\mu_1 \leq \mu_2 \leq \mu_3 = \mu_3^{\aleph_0} \leq \mu_4 = \mu_4^{\aleph_0}$ *be uncountable regular cardinals,* $\mu_4 < \mu_5 = \mu_5^{<\mu_4} \leq 2^{\mu_3}$. *Then, there is a ccc poset forcing*



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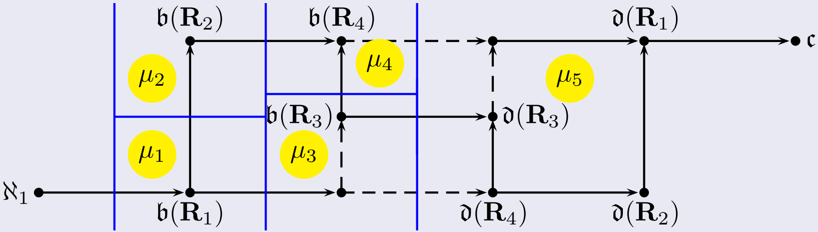

(Goldstern & Kellner & Shelah 2017–2019) Can obtain such a ccc poset under GCH.

Natural attempt

Construct a FS it. of length μ_5 alternating:

1. $\mathbb{LOC}^N$ with $|N| < \mu_1$,
2. $(\text{random})^N$ with $|N| < \mu_2$,
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via book-keeping to get, for $i = 1, 2, 3, 4$,

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Hence $\mu_i \leq \mathfrak{b}(\mathbf{R}_i)$ and $\mathfrak{d}(\mathbf{R}_i) \leq \mu_5$ (actually $\mathfrak{c} \leq \mu_5$).

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The iterands are, for $i = 1, 2, 4$,

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Hence $\mathfrak{b}(\mathbf{R}_i) \leq \mu_i$ and $\mu_5 \leq \mathfrak{d}(\mathbf{R}_i)$.

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$\mathbb{E}$ *is* $\aleph_1$-$\mathbf{R}_3$-*good.*

However

Theorem (Pawlikowski 1992)

There is a proper ω^ω-bounding poset forcing that $\mathbb{E}^V$ adds a dominating real.

Filter-linkedness

Definition (M. 2018arxiv–2019pub)

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Lemma (M. 2018–2019)

If $\mathbb{P}$ is ccc then $Q \subseteq \mathbb{P}$ is uf-linked iff it is Fr-linked.

Filter-linkedness (cont.)

For $x, y \in \omega^\omega$ denote $x \leq_n y$ iff $(\forall i \geq n)\, x(i) \leq y(i)$.

Lemma

If $Q \subseteq \mathbb{P}$ is Fr-linked and $\dot{y} \in \omega^\omega$ then

$$(\exists y \in \omega^\omega)(\forall x \in \omega^\omega)(\forall n < \omega)\, x \not\leq_n y \Rightarrow (\forall p \in Q)\, p \not\Vdash x \leq_n \dot{y}.$$

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GMS and GKS posets for the left side of Cichoń's diagram are θ-Fr-Knaster for any regular $\mu_3 \leq \theta \leq \mu_5$. In particular $\mathrm{LCU}_{\mathbf{R}_3}(\mathbb{P}, \theta)$ holds.

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Order: $(s', \varphi') \leq (s, \varphi)$ iff $s \subseteq s'$, $(\forall i)\, \varphi(i) \subseteq \varphi'(i)$ and $s'(i) \notin \varphi(i)$ for all $i \in |s'| \smallsetminus |s|$.

Eventually different real forcing

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Clearly $\mathbb{E}$ is σ-centered and the generic real $e := \bigcup\{s : \exists \varphi((s, \varphi) \in G)\}$ is eventually different over the ground model (so it increases $\mathrm{non}(\mathcal{M})$).

Examples of μ-uf-linked posets

Lemma (ess. Miller 1981)

$E_{(t,m)} := \{(s,\varphi) \in \mathbb{E} : s = t,\ (\forall i)\, |\varphi(i)| \leq m\}$ is uf-linked.
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Any complete Boolean algebra with a strictly-positive countable-additive measure is σ-Fr-linked.
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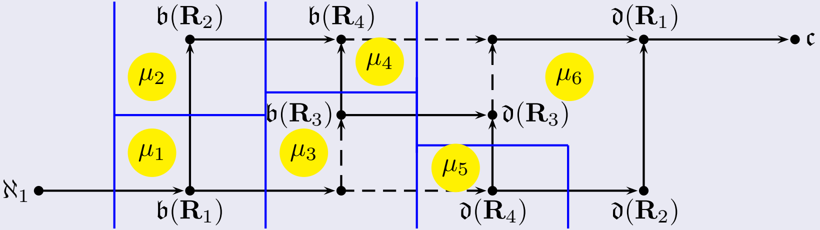
Lemma

If $|\mathbb{P}| \leq \mu$ then $\mathbb{P}$ is μ-uf-linked.
(b/c singletons are uf-linked)

Theorem (Brendle & Cardona & M. 2018)

Let $\aleph_1 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \leq \mu_5$ be regular and $\mu_5 \leq \mu_6 = \mu_6^{<\mu_3}$. Then, there is a ccc poset forcing



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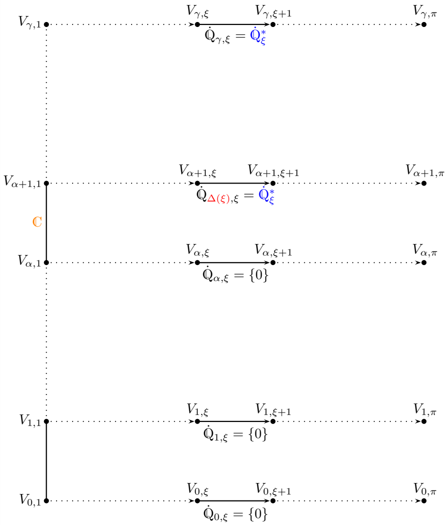

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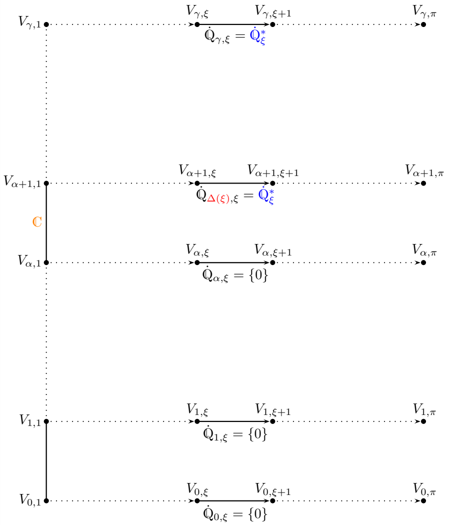

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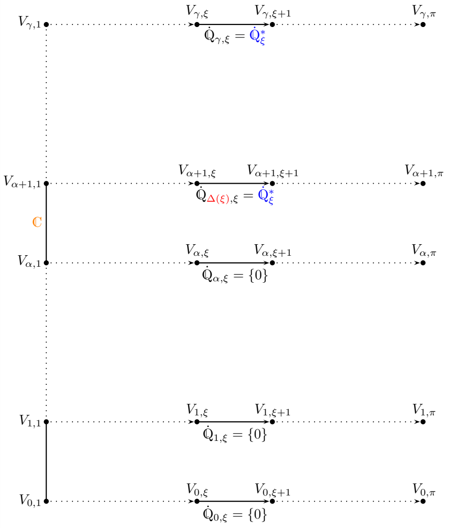

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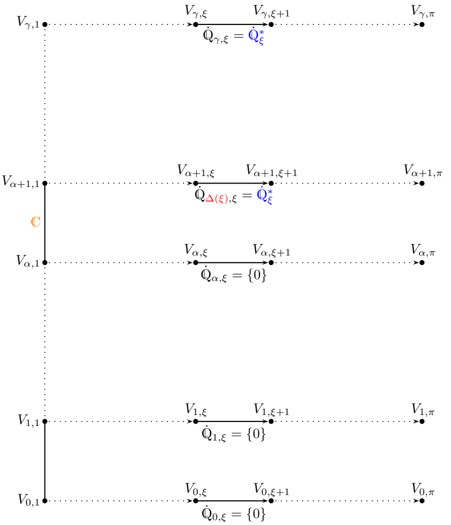

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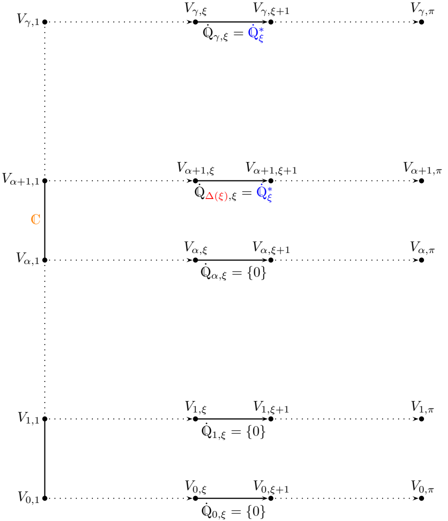

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Assume $\mathrm{cf}(\gamma) \geq \omega_1$, $\mathbb{P}_{\gamma,\pi}$ is ccc and $\mathbf{R} = \langle X, Y, R\rangle$ is Polish.

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1. $(\forall y \in Y \cap V_{\alpha,\pi})\, \neg(c_\alpha R y)$.
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Proof of the theorem

First force with $\mathbb{C}_{\mu_6}$, so $\mathrm{LCU}_{\mathbf{R}_i}(\mathbb{C}, \theta)$ for any regular $\aleph_1 \leq \theta \leq \mu_6$.

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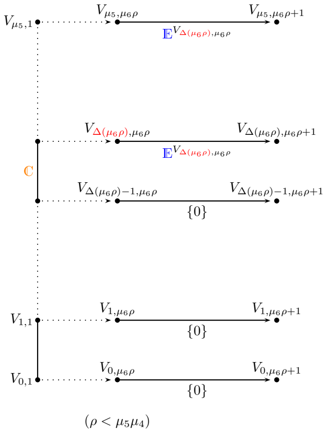

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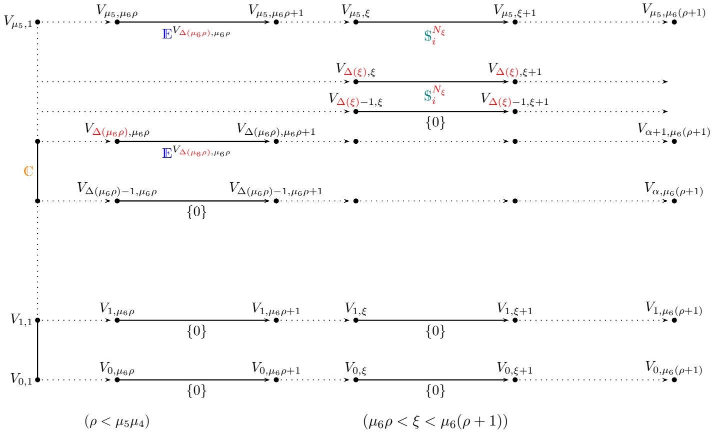

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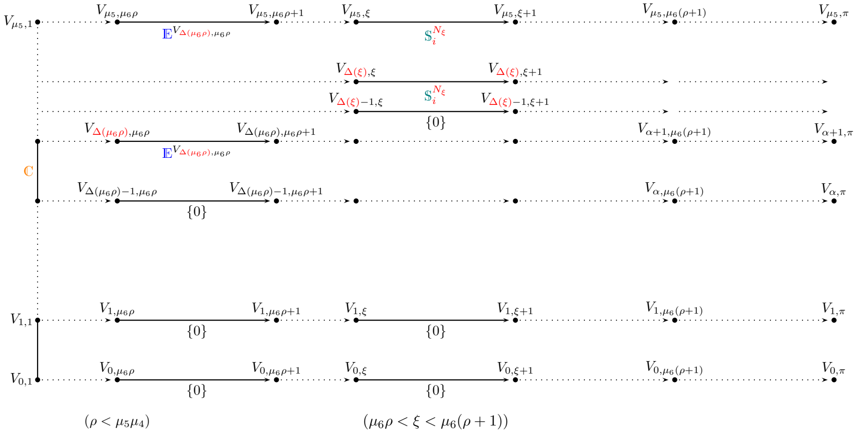

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Δ and N_ξ are constructed so that:

1. For $i = 1, 2, 3$, $\mathrm{COB}_{\mathbf{R}_i}(\mathbb{P}, S_i)$ where $\mu_i \leq \mathrm{cp}(S_i) \leq \mathrm{cf}(S_i) \leq |S_i| = \mu_6$, so $\mu_i \leq \mathfrak{b}(\mathbf{R}_i)$, $\mathfrak{d}(\mathbf{R}_i) \leq \mu_6$.

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We are done if we show that the matrix is μ_3-uf-Knaster.

κ-uf-Knaster matrices

Theorem (Brendle & Cardona & M. 2018)

Let $\kappa \geq \aleph_1$ be regular. If $\mathbb{P}$ is a simple matrix iteration such that $V_{\Delta(\xi),\xi} \models$ "$\mathbb{Q}^*_\xi$ is $<\kappa$-uf-linked" for any $0 < \xi < \pi$, then $\mathbb{P}$ is κ-uf-Knaster.

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So $V_{\gamma,\xi} \models$"$\mathbb{Q}^*_\xi$ is $< \kappa$-Fin-cc", hence κ-cc. Thus $\mathbb{P}$ is κ-cc.

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Let $\kappa \geq \aleph_1$ be regular. If $\mathbb{P}$ is a simple matrix iteration such that $V_{\Delta(\xi),\xi} \models$ "$\dot{\mathbb{Q}}^*_\xi$ is $<\kappa$-uf-linked" for any $0 < \xi < \pi$, then $\mathbb{P}$ is κ-uf-Knaster.

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($Q \subseteq \mathbb{Q}$ is Fin-cc if $(\forall A \subseteq Q)$ (A antichain in $\mathbb{P}$ $\Rightarrow$ $|A| < \aleph_0$))

So $V_{\gamma,\xi} \models$"$\dot{\mathbb{Q}}^*_\xi$ is $<\kappa$-Fin-cc", hence κ-cc. Thus $\mathbb{P}$ is κ-cc.

For each $0 < \xi < \pi$ there is some cardinal $\theta_\xi < \kappa$ and $\mathbb{P}_{\Delta(\xi),\xi}$-names $\langle \dot{Q}_{\xi,\zeta} : \zeta < \theta_\xi \rangle$ of uf-linked subsets of $\dot{\mathbb{Q}}^*$ s.t. $\dot{\mathbb{Q}}^*_\xi = \bigcup_{\zeta<\theta_\xi} \dot{Q}_{\xi,\zeta}$.

Main Lemma

Wlog $p \in \mathbb{P}$ iff $(\forall \xi \in \mathrm{supp} p)\ p(\xi) \in \dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_{\Delta(\xi),\xi}$-name, and there is some $f_p \in \prod_{\xi \in \mathrm{supp} p} \theta_\xi$ s.t. $\Vdash_{\Delta(\xi),\xi} p(\xi) \in \dot{Q}_{\xi, f_p(\xi)}$.

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Main Lemma

If D is a non-principal ultrafilter on ω and $\langle p_n : n < \omega \rangle \subseteq \mathbb{P}$ is uniform, then, in $V^{\mathbb{P}}$, there is some ultrafilter $D^* \supseteq D$ such that $\{n < \omega : p_n \in G_{\mathbb{P}}\} \in D^*$.

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$\mathbb{P}$ forces that $\{n < \omega : p_n \in \dot{G}\} \in D^*$ for some uf $D^* \supseteq D$,

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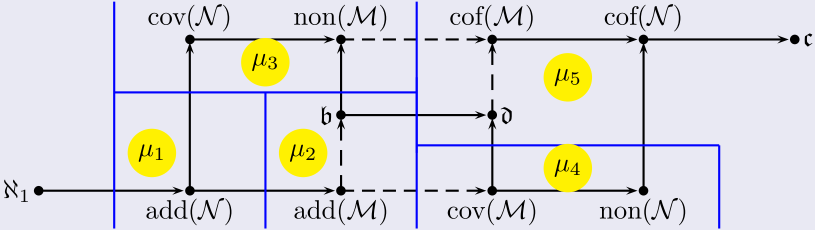
Hence B is D-linked.

Another application

Theorem (Brendle & Cardona & M. 2018)

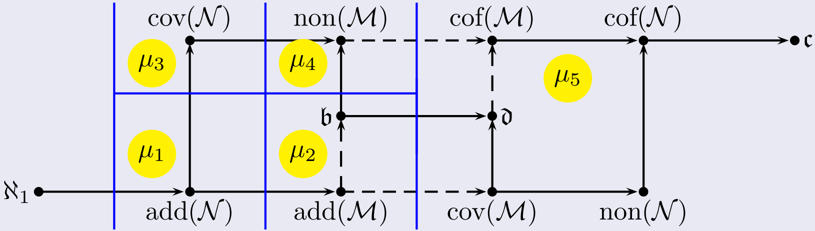
Let $\aleph_1 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4$ be regular cardinals, $\mu_4 \leq \mu_5 = \mu_5^{<\mu_2}$. Then, there is a ccc poset forcing



The other left side

Theorem (Kellner & Tănasie & Shelah 2018arxiv–2019pub)

Let $\aleph_1 \leq \mu_1 \leq \mu_2 = \mu_2^{<\mu_2} < \mu_3 \leq \mu_4$ *be regular cardinals,* $\mu_4^{\aleph_0} < \mu_5 = \mu_5^{<\mu_4}$*, and* $(\forall \nu < \mu_3)\, \nu^{\aleph_0} < \mu_3$*. Then, there is a ccc poset forcing*



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$$xR_{\mathrm{md}}h \text{ iff } (\exists n)(\forall i \geq n)(\forall F \in [A]^{<\aleph_0})\, [n, h(n,F)) \smallsetminus \bigcup F \nsubseteq x$$

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If A is an $\mathbf{R}_{\mathrm{md}}(A)$-unbounded a.d. family, then A is mad.

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Theorem (Brendle & Cardona & M. 2018)

If $\kappa \geq \aleph_1$ is regular, $\mathbb{P}$ is κ-Fr-Knaster and A is κ-$\mathbf{R}_{\mathrm{md}}(A)$-LCU, then $\mathbb{P}$ forces that A is still κ-$\mathbf{R}_{\mathrm{md}}(A)$-LCU.