Higher cardinal invariants

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Outline

1 Bounding and dominating

2 Splitting and reaping

Bounding and dominating

Definition

Let $\kappa \geq \omega$ *be a regular cardinal. Let* $f, g \in \kappa^\kappa$. $f \leq^* g$ *means that* $|\{\alpha < \kappa : g(\alpha) < f(\alpha)\}| < \kappa$

Definition

We say that $F \subseteq \kappa^\kappa$ *is* $*$***-unbounded*** *if* $\neg \exists g \in \kappa^\kappa \forall f \in F\, [f \leq^* g]$.

Definition

$\mathfrak{b}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \wedge F$ *is* $*$ *-unbounded*$\}$.

Definition

We say that $F \subseteq \kappa^{\kappa}$ is $*$***-dominating*** *if* $\forall g \in \kappa^{\kappa} \exists f \in F \left[g \leq^* f \right]$

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$\mathfrak{d}(\kappa) = \min \{|F| : F \subseteq \kappa^{\kappa}$ *and* F *is* $*$ *-dominating*$\}$.

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Definition

$\mathfrak{d}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa$ *and F is $*$-dominating*$\}$.

Theorem

For any regular $\kappa \geq \omega$, $\kappa^+ \leq \mathrm{cf}(\mathfrak{b}(\kappa)) = \mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$

- These are the only relations between $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ that are provable in ZFC (Hechler for ω; Cummings and Shelah for $\kappa > \omega$).

- When $\kappa > \omega$, we can also use the club filter.

Definition

Let $\kappa > \omega$ be a regular cardinal. $f \leq_{\mathrm{cl}} g$ means that $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary. For $F \subseteq \kappa^\kappa$, we say that:

- *F is cl-**unbounded** if $\neg\exists g \in \kappa^\kappa \forall f \in F\, [f \leq_{\mathrm{cl}} g]$, and*
- *F is cl-**dominating** if $\forall g \in \kappa^\kappa \exists f \in F\, [g \leq_{\mathrm{cl}} f]$.*

Definition

We define

$$\mathfrak{b}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \wedge F \text{ is cl-unbounded}\},$$
$$\mathfrak{d}_{\mathrm{cl}}(\kappa) = \min\left\{|F| : F \subseteq \kappa^\kappa \text{ and } F \text{ is cl-dominating}\right\}.$$

Theorem (Cummings and Shelah)

For every regular cardinal $\kappa > \omega$, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\text{cl}}(\kappa)$.

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If $\kappa \geq \beth_\omega$ *is regular, then* $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$.

Question

Does $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$, *for every regular uncountable* κ? *In particular, does* $\mathfrak{d}(\omega_1) = \mathfrak{d}_{\mathrm{cl}}(\omega_1)$?

Splitting and reaping

Definition

Let $\kappa \geq \omega$ be regular.

- *For $A, B \in \mathcal{P}(\kappa)$, A **splits** B if $|B \cap A| = |B \cap (\kappa \setminus A)| = \kappa$.*
- *$F \subseteq \mathcal{P}(\kappa)$ is called a **splitting family** if $\forall B \in [\kappa]^{\kappa} \exists A \in F\, [A \text{ splits} B]$.*

$$\mathfrak{s}(\kappa) = \min\{|F| : F \subseteq \mathcal{P}(\kappa) \wedge F \text{ is a splitting family}\};$$

Theorem (Solomon)

$\omega_1 \leq \mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$.

Theorem (Suzuki)

For a regular $\kappa > \omega$, $\mathfrak{s}(\kappa) \geq \kappa$ *iff* κ *is strongly inaccessible and* $\mathfrak{s}(\kappa) \geq \kappa^+$ *iff* κ *is weakly compact.*

- So if κ is not weakly compact, then $\mathfrak{s}(\kappa) < \kappa^+ \leq \mathfrak{b}(\kappa)$.

Theorem (Suzuki)

For a regular $\kappa > \omega$, $\mathfrak{s}(\kappa) \geq \kappa$ iff κ is strongly inaccessible and $\mathfrak{s}(\kappa) \geq \kappa^+$ iff κ is weakly compact.

- So if κ is not weakly compact, then $\mathfrak{s}(\kappa) < \kappa^+ \leq \mathfrak{b}(\kappa)$.

Theorem (Zapletal)

If it is consistent to have a regular uncountable cardinal κ such that $\mathfrak{s}(\kappa) \geq \kappa^{++}$, then it is also consistent that there is a κ with $o(\kappa) \geq \kappa^{++}$.

Theorem (Ben-Neria and Gitik)

If $o(\kappa) = \kappa^{++}$, then there is a forcing extension in which $\mathfrak{s}(\kappa) = \kappa^{++}$.

- However κ does not remain measurable in their model.

Question

What is the consistency strength of the statement that κ is a measurable cardinal and $\mathfrak{s}(\kappa) = \kappa^{++}$?

- If κ is supercompact, it is not difficult to produce a model where κ remains supercompact and $\mathfrak{s}(\kappa) = \kappa^{++}$.

- $\mathfrak{s}(\omega)$ and $\mathfrak{b}(\omega)$ are independent.

Theorem (Baumgartner and Dordal)

It is consistent to have $\mathfrak{s}(\omega) < \mathfrak{b}(\omega)$.

Theorem (Shelah)

It is consistent to have $\omega_1 = \mathfrak{b}(\omega) < \mathfrak{s}(\omega) = \omega_2$.

- It turns out the ω is the *only regular cardinal* for which the statement $\mathfrak{b}(\kappa) < \mathfrak{s}(\kappa)$ is consistent.

Theorem (R. and Shelah[1])

For any regular uncountable cardinal κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

- The proof of this is surprisingly elementary, relying on two standard facts.

- Recall the Katětov order on ideals.

Definition

*Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on κ. $\mathcal{I}$ is **Katětov below** $\mathcal{J}$ if there is a function $f_* : \kappa \to \kappa$ such that $\forall D \in \mathcal{I} \left[f_*^{-1}(D) \in \mathcal{J} \right]$.*

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- The main point in the proof of $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ is that if M is a model of (a sufficient fragment of) set theory, then $\mathrm{NS}_\kappa \cap M$ is Katětov below every κ-complete maximal ideal over $M \cap \mathcal{P}(\kappa)$.

Lemma

Let $\kappa > \omega$ be a regular cardinal and $M \prec H(\theta)$, where θ is a sufficiently large regular cardinal. If there is a set $B \in [\kappa]^\kappa$ such that $B \subseteq^* C$ for every club $C \in M$ of κ, then $M \cap \kappa^\kappa$ is bounded.

Proof.

Let $\langle \beta_\xi : \xi < \kappa \rangle$ enumerate B in strictly increasing order. Let $h : \kappa \to \kappa$ be defined by $h(\xi) = \beta_{\xi+1}$. We will check that h dominates all of $M \cap \kappa^\kappa$. Consider any $f \in M \cap \kappa^\kappa$. Then $C_f = \{\xi < \kappa : \xi \text{ is closed under } f\} \in M$ and it is a club in κ. So there exists $\delta < \kappa$ such that $B \setminus \delta \subseteq C_f$. We will check that for any $\alpha \geq \beta_\delta$, $h(\alpha) > f(\alpha)$. We have $\delta \leq \beta_\delta \leq \alpha < \alpha + 1 \leq \beta_{\alpha+1}$. So $\beta_{\alpha+1} \in C_f$, and so $f(\alpha) < \beta_{\alpha+1} = h(\alpha)$. ⊣

- Let $\kappa > \omega$ be regular and suppose that $\kappa^+ < \mathfrak{s}(\kappa)$.
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- Let $M \prec H(\theta)$ be such that $\lambda \subseteq M$ and $|M| = \lambda$ (θ is any sufficiently large regular cardinal). Since $M \cap \mathcal{P}(\kappa)$ is not a splitting family, there exists $A_* \in [\kappa]^\kappa$ which **decides** every $x \in M \cap \mathcal{P}(\kappa)$ (i.e. either $A_* \subseteq^* x$ or $A_* \subseteq^* (\kappa \setminus x)$, where $X \subseteq^* Y$ means $|X \setminus Y| < \kappa$).
- Define D to be $\{x \in \mathcal{P}(\kappa) : A_* \subseteq^* x\}$.

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- Define D to be $\{x \in \mathcal{P}(\kappa) : A_* \subseteq^* x\}$.
- For any $f, g \in M \cap \kappa^\kappa$, define $f \sim_D g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$.
- This is an equivalence relation on $M \cap \kappa^\kappa$.
- For $f \in M \cap \kappa^\kappa$, let $[f]_D = \{g \in M \cap \kappa^\kappa : f \sim_D g\}$.
- For $f, g \in M \cap \kappa^\kappa$, define $[f]_D <_D [g]_D$ iff $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$.
- Let $L = \{[f]_D : f \in M \cap \kappa^\kappa\}$.

- $\langle L, <_D \rangle$ is a linear order.
- In fact, $\langle L, <_D \rangle$ is a well-order because D is a κ-complete filter on κ (and because M is closed under various operations).
- But we only need to know that the constant functions $\{[c_\alpha]_D : \alpha < \kappa\}$ have a least upper bound in L, where c_α is the function $\delta \mapsto \alpha$.

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Lemma

Suppose $f_* \in M \cap \kappa^\kappa$ is such that $[f_*]_D \in L$ is a least upper bound of $\{[c_\alpha]_D : \alpha < \kappa\}$ in $\langle L, <_D \rangle$. Then for any $C \in M$ which is a club in κ, $f_*^{-1}(C) \in D$.

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Lemma

Suppose $f_* \in M \cap \kappa^\kappa$ is such that $[f_*]_D \in L$ is a least upper bound of $\{[c_\alpha]_D : \alpha < \kappa\}$ in $\langle L, <_D \rangle$. Then for any $C \in M$ which is a club in κ, $f_*^{-1}(C) \in D$.

- Let $B = f_*'' A_*$. Then $B \in [\kappa]^\kappa$ because $[f_*]_D$ bounds all constant functions.
- Also if $C \in M$ is any club of κ, then $f_*'' A_* \subseteq^* C$.
- It follows that $M \cap \kappa^\kappa$ is bounded.

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- So for any cardinal λ such that $\kappa < \lambda < \mathfrak{s}(\kappa)$, if $\mathcal{F} \subseteq \kappa^\kappa$ with $|\mathcal{F}| \leq \lambda$, then $\mathcal{F}$ is bounded.
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- R. and Shelah also proved that for a supercompact κ, it is consistent to have $\kappa^+ = \mathfrak{s}(\kappa) < \mathfrak{b}(\kappa) = \kappa^{++}$ (unpublished).
- Do a $< \kappa$-support iteration $\langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha < \kappa^{++} \rangle$ so that if $\alpha < \kappa^+$, then $\mathring{\mathbb{Q}}_\alpha$ is the forcing for adding a Cohen subset of κ, while if $\kappa^+ \leq \alpha < \kappa^{++}$, then $\mathring{\mathbb{Q}}_\alpha$ is the forcing for adding a dominating function from κ to κ

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- Exercise: check that the first κ^+ Cohen subsets remain a splitting family in the end.

Question

Is it possible to have $\kappa^+ = \mathfrak{s}(\kappa) < \mathfrak{b}(\kappa) < 2^\kappa$?

- $\mathfrak{b}(\omega)$ and $\mathfrak{d}(\omega)$ are dual to each other.
- The dual of $\mathfrak{s}(\omega)$ is $\mathfrak{r}(\omega)$.

Definition

For a family $F \subseteq [\kappa]^\kappa$ and a set $B \in \mathcal{P}(\kappa)$, B is said to ***reap*** *F if for every $A \in F$, $|A \cap B| = |A \cap (\kappa \setminus B)| = \kappa$. We say that $F \subseteq [\kappa]^\kappa$ is* ***unreaped*** *if there is no $B \in \mathcal{P}(\kappa)$ that reaps F.*

- $F \subseteq [\kappa]^\kappa$ is unreaped iff each $B \in \mathcal{P}(\kappa)$ is **decided** by some member of F.

Definition

$\mathfrak{r}(\kappa) = \min \{|F| : F \subseteq [\kappa]^\kappa$ *and* F *is unreaped*$\}$.

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Definition

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- The proof of $\mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$ dualizes to the proof of $\mathfrak{b}(\omega) \leq \mathfrak{r}(\omega)$.
- Also $\mathfrak{r}(\omega)$ and $\mathfrak{d}(\omega)$ are independent.
- Not clear if there is a good theory of duality at uncountable regular cardinals too.
- For example, Suzuki's theorem says that $\mathfrak{s}(\kappa)$ is small unless κ is weakly compact.
- So we might expect that $\mathfrak{r}(\kappa)$ is large below the first weakly compact cardinal (will be taken up in the next tutorial).

- The proof that for all $\kappa > \omega$, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ does not dualize.
- But the theorem does have a partial dual:

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Theorem (R. + Shelah [2])

For all regular cardinals $\kappa \geq \beth_{\omega}$, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

- So for sufficiently large κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ provably in ZFC.

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- So for sufficiently large κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ provably in ZFC.
- The proof of this is an application of PCF theory to cardinal invariants.
- We use the revised GCH.

Definition

Let κ and λ be cardinals. Define $\lambda^{[\kappa]}$ to be

$$\min\left\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq\kappa} \text{ and } \forall u \in [\lambda]^{\kappa} \exists \mathcal{P}_0 \subseteq \mathcal{P}\left[|\mathcal{P}_0| < \kappa \text{ and } u = \bigcup \mathcal{P}_0\right]\right\}.$$

The operation $\lambda^{[\kappa]}$ is sometimes referred to as the ***weak power****.*

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*The operation $\lambda^{[\kappa]}$ is sometimes referred to as the **weak power**.*

- Easy exercise: GCH is equivalent to the statement that for all regular cardinals $\kappa < \lambda$, $\lambda^{[\kappa]} = \lambda$.
- The revised GCH, which is a theorem of ZFC says that for “lots of pairs” of regular cardinals we have $\lambda^{[\kappa]} = \lambda$.

Theorem (Shelah; The Revised GCH)

If θ is a strong limit uncountable cardinal, then for every $\lambda \geq \theta$, there exists $\sigma < \theta$ such that for every $\sigma \leq \kappa < \theta$, $\lambda^{[\kappa]} = \lambda$.

Corollary

Let $\mu \geq \beth_\omega$ be any cardinal. There exists an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\mu]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and for each $u \in [\mu]^\theta$, there exists $v \in \mathcal{P}$ with the property that $v \subseteq u$ and $|v| \geq \aleph_0$.

- This corollary is used with $\mu = \mathfrak{r}(\kappa)$.

- Actually the proof breaks into two cases and the revised GCH is only needed in one of the cases.

Definition

Let $E_2 \subseteq E_1$ both be clubs in κ. Define
$\mathrm{set}\,(E_2, E_1) = \bigcup \{[\xi, \mathrm{Next}_{E_1}(\xi)) : \xi \in E_2\}$.

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- The two cases are:
  1. There is an unreaped family $\mathcal{F} \subseteq [\kappa]^{\kappa}$ of minimal cardinality with the property that there is a club $E_1 \subseteq \kappa$ such that for each club $E_2 \subseteq E_1$, there exists $B \in F$ with $B \subseteq^* \mathrm{set}(E_2, E_1)$.

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  2. For every unreaped family $\mathcal{F} \subseteq [\kappa]^{\kappa}$ of minimal cardinality, for each club $E_1 \subseteq \kappa$, there exist a club $E_2 \subseteq E_1$ and a $B \in F$ such that $B \subseteq^* (\kappa \setminus \mathrm{set}(E_2, E_1))$.
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- The revised GCH is only needed in Case 2.
- I do not know if Case 2 can occur when (for example) $\kappa = \aleph_1$.

Question

Is $\mathfrak{d}(\aleph_1) \leq \mathfrak{r}(\aleph_1)$ provable? Is $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ provable for all regular $\kappa < \beth_\omega$?

- If $\mathfrak{d}(\aleph_1) > \mathfrak{r}(\aleph_1)$, then the corollary from the previous slide must fail for $\mu = \mathfrak{r}(\aleph_1)$.
- This is known to imply the existence of large cardinals (e.g. there is a κ with $o(\kappa) = \kappa^+$).
- There is an even more basic question.

Question

Is it consistent (relative to large cardinals) that $\mathfrak{r}(\omega_1) < 2^{\aleph_1}$?

- This is related to an old question of Kunen about bases for uniform ultrafilters (will be taken up in the next tutorial).

Bibliography

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