Higher cardinal invariants

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Outline

1. Ultrafilters and almost disjoint families

2. When is the reaping number small?

Definition

Let $\kappa \geq \omega$ be regular. Let $\mathcal{U}$ be an ultrafilter on κ. We say that:

- *$\mathcal{U}$ is **uniform** if every element of $\mathcal{U}$ has cardinality κ;*
- *$F \subseteq \mathcal{P}(\kappa)$ is a **base for** $\mathcal{U}$ if $\mathcal{U} = \{B \subseteq \kappa : \exists A \in F\,[A \subseteq B]\}$.*

Definition

$\mathfrak{u}(\kappa) = \min\{|F| : F$ *is a base for a uniform ultrafilter on* $\kappa\}$.

- Clearly $\mathfrak{r}(\kappa) \leq \mathfrak{u}(\kappa)$ ($F \subseteq [\kappa]^{\kappa}$ and F needs to decide every subset of κ to generate a uniform ultrafilter).
- $\mathfrak{u}(\omega)$ and $\mathfrak{s}(\omega)$ are independent.
- However for $\kappa > \omega$, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{r}(\kappa)$.

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- $\mathfrak{u}(\omega)$ and $\mathfrak{s}(\omega)$ are independent.
- However for $\kappa > \omega$, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{r}(\kappa)$.

Question (Kunen)

Is it consistent that $\mathfrak{u}(\omega_1) < 2^{\aleph_1}$*?*

Theorem (Carlson, 1980s unpublished)

If κ *is supercompact, then* $\mathfrak{u}(\kappa) < 2^{\kappa}$ *is consistent.*

- How about getting $\mathfrak{u}(\kappa) < 2^{\kappa}$ at smaller more accessible cardinals?
- R. + Shelah showed recently that it is possible to do this for many accessible cardinals, assuming large cardinals.

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Theorem (R. + Shelah [2], 2018)

It is consistent relative to a measurable cardinal that there is a uniform ultrafilter on the reals which is generated by fewer than $2^{2^{\aleph_0}}$ many sets.

Theorem (R. + Shelah [2], 2018)

Assume that there is a supercompact cardinal. Then there is a forcing extension in which $\mathfrak{u}(\aleph_{\omega+1}) < 2^{\aleph_{\omega+1}}$.

- The crucial ingredient used in these proofs is the notion of an indecomposable filter.

Definition

*Let κ and λ be infinite cardinals. A filter $\mathcal{F}$ on λ is said to be κ-**indecomposable** if whenever $\langle Y_\xi : \xi < \kappa \rangle$ is a partition of λ – i.e. $\lambda = \bigcup_{\xi<\kappa} Y_\xi$ and $\forall \zeta < \xi < \kappa \, [Y_\zeta \cap Y_\xi = 0]$ – then there exists $T \subseteq \kappa$ such that $|T| < \kappa$ and $\bigcup_{\xi \in T} Y_\xi \in \mathcal{F}$.*

- Note that $\aleph_0$-indecomposable is the same as countably complete.
- If $\mathcal{F}$ is a λ-complete ultrafilter on λ, then it is κ-indecomposable for any $\aleph_0 \leq \kappa < \lambda$.

Definition

*Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ be a forcing notion. We say that $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ has a $(\lambda, \kappa, \mu, \mathcal{D})$-**filtration** if there exists a sequence $\langle \mathbb{P}_\alpha : \alpha < \mu \rangle$ satisfying the following:*

1. *λ, κ, and μ are infinite cardinals satisfying $\lambda < \mathrm{cf}(\mu) < \kappa < \mu$;*
2. *μ is a strong limit cardinal and $\lambda^{<\lambda} = \lambda$;*
3. *$\mathcal{D}$ is a uniform $\mathrm{cf}(\mu)$-indecomposable filter on κ;*
4. *$\mathbb{P}$ is λ^+-c.c. and $\forall p \in \mathbb{P} \exists \alpha < \mu \, [p \in \mathbb{P}_\alpha]$;*
5. *for each $\alpha < \mu$, $\mathbb{P}_\alpha \subseteq_{\mathrm{c}} \mathbb{P}$, and $\forall \xi < \alpha \, [\mathbb{P}_\xi \subseteq \mathbb{P}_\alpha]$;*
6. *for each $\alpha < \mu$, $|\mathbb{P}_\alpha| < \mu$.*

- Observe that there is no connection between $\mathbb{P}$ and the filter $\mathcal{D}$ – i.e. we only need the existence of some uniform $\mathrm{cf}(\mu)$-indecomposable filter on κ.

Theorem

Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ *be a forcing notion. Assume that* λ, κ, μ*, and* $\mathcal{D}$ *are so that* $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ *has a* $(\lambda, \kappa, \mu, \mathcal{D})$*-filtration. Assume moreover that* $\mathrm{cf}(\kappa) = \kappa$*. Then* $\mathbb{P}$ *forces that every uniform ultrafilter on* κ *that extends* $\mathcal{D}$ *is generated by a set of size at most* μ*. In particular,* $\mathbb{P}$ *forces that* $\mathfrak{u}(\kappa) \leq \mu$*.*

- Several posets of the form $\mathrm{Fn}(I, J, \chi)$ as well as products of such posets have a $(\lambda, \kappa, \mu, \mathcal{D})$-filtration.

Lemma

Suppose that λ, κ, μ*, and* $\mathcal{D}$ *satisfy the first three conditions in the definition of a* $(\lambda, \kappa, \mu, \mathcal{D})$*-filtration. Then* $\mathrm{Fn}(\mu \times \lambda, 2, \lambda)$ *has a* $(\lambda, \kappa, \mu, \mathcal{D})$*-filtration.*

- The following theorem is needed for the case when $\kappa = \aleph_{\omega+1}$, and it uses a supercompact cardinal.

Theorem (Ben-David and Magidor)

Assume that there is a supercompact cardinal. There is a forcing extension in which GCH *holds and there is a uniform ultrafilter on* $\aleph_{\omega+1}$ *which is* $\aleph_n$*-indecomposable for all* $0 < n < \omega$.

Theorem

Assume that there is a supercompact cardinal. Then there is a forcing extension in which $\mathfrak{u}(\aleph_{\omega+1}) < 2^{\aleph_{\omega+1}}$.

Proof.

By the result of Ben-David and Magidor we can pass to a forcing extension $\mathbf{V}'$ in which GCH holds and there exists a uniform ultrafilter $\mathcal{D}$ on $\aleph_{\omega+1}$ which is $\aleph_n$-indecomposable for all $0 < n < \omega$. Working in $\mathbf{V}'$, put $\kappa = \aleph_{\omega+1}$ and choose $\lambda = \aleph_0$ and $\mu = \aleph_{\omega_1}$. Then since GCH holds in $\mathbf{V}'$, (1)–(3) of the definition of a $(\lambda, \kappa, \mu, \mathcal{D})$-filtration are satisfied. So $\mathbb{P} = \mathrm{Fn}(\mu \times \lambda, 2, \lambda)$ has a $(\lambda, \kappa, \mu, \mathcal{D})$-filtration. Let G be $(\mathbf{V}', \mathbb{P})$-generic. By standard arguments, $2^{\lambda} = \mu$ in $\mathbf{V}'[G]$. By the main theorem, $\mathfrak{u}(\kappa) \leq \mu$ holds in $\mathbf{V}'[G]$. By the fact the all cofinalities and cardinals are preserved between $\mathbf{V}'$ and $\mathbf{V}'[G]$, $\aleph_{\omega+1} = \kappa$, $\mathfrak{u}(\kappa) \leq \mu$, and $2^{\kappa} = \left(2^{\lambda}\right)^{\kappa} = \mu^{\kappa} \geq \mu^{\mathrm{cf}(\mu)} > \mu$ in $\mathbf{V}'[G]$. $\dashv$

- $2^{\aleph_{\omega+1}} > \aleph_{\omega_1}$ holds in this model and we do not know how to get a smaller gap.

Question

Is it possible to produce models where $\mathfrak{u}(\aleph_{\omega+1}) = \aleph_{\omega+2} < \aleph_{\omega+3} = 2^{\aleph_{\omega+1}}$*?*

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- Another feature of this model is that $2^{\aleph_0} = \aleph_{\omega_1}$.
- Actually we did not have to choose $\lambda = \aleph_0$. We could have chosen $\lambda = \aleph_n$, for some $n < \omega$ and $\mu = \aleph_{(\omega_{n+1})}$.
- Then we would have GCH below $\aleph_n$, $2^{\aleph_n} = \aleph_{(\omega_{n+1})}$ and $2^{\aleph_{\omega+1}} = \aleph_{(\omega_{n+1}+1)}$.

Question

Is it consistent to have $\aleph_\omega$ *be a strong limit and* $\mathfrak{u}(\aleph_{\omega+1}) < 2^{\aleph_{\omega+1}}$*?*

- For ultrafilters on the reals we need one more fact.

Lemma

Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ *and* $\langle \mathbb{R}, \leq_{\mathbb{R}}, 1_{\mathbb{R}} \rangle$ *be forcing notions. Assume that* λ, κ, μ, *and* $\mathcal{D}$ *are so that* $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ *has a* $(\lambda, \kappa, \mu, \mathcal{D})$*-filtration. If* $|\mathbb{R}| < \mu$ *and*

$$\Vdash_{\mathbb{P}} \text{“}\check{\mathbb{R}} \text{ is } \check{\lambda}^{+}\text{-c.c.”},$$

then $\mathbb{P} \times \mathbb{R}$ *also has a* $(\lambda, \kappa, \mu, \mathcal{D})$*-filtration.*

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then $\mathbb{P} \times \mathbb{R}$ *also has a* $(\lambda, \kappa, \mu, \mathcal{D})$*-filtration.*

- Starting with a measurable cardinal κ, and applying the above lemma (to a product of the form $\mathrm{Fn}(\mu \times \lambda, 2, \lambda) \times \mathrm{Fn}(\kappa \times \aleph_0, 2, \aleph_0)$) together with the main theorem, we get a model where $\mathfrak{u}(2^{\aleph_0}) < 2^{2^{\aleph_0}}$.

- In the resulting model $2^{\aleph_0} = \kappa$ and it is weakly inaccessible.
- And $2^{\kappa} > \kappa^{+\aleph_2}$. This is the smallest gap we are able to get.

Question

Is it consistent to have $2^{\aleph_0} = \kappa$ *regular and* $\mathfrak{u}(2^{\aleph_0}) = \kappa^{+} < \kappa^{++} = 2^{2^{\aleph_0}}$*?*

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Question

Is it consistent to have $2^{\aleph_0} = \kappa$ regular and $\mathfrak{u}(2^{\aleph_0}) = \kappa^{+} < \kappa^{++} = 2^{2^{\aleph_0}}$?

- It is also possible to combine the two theorems to get the following.

Corollary

Assume that there is a supercompact cardinal. Then there is a forcing extension in which $2^{\aleph_0} = \aleph_{\omega+1}$ and $\mathfrak{u}(\aleph_{\omega+1}) < 2^{\aleph_{\omega+1}}$.

Definition

Let $\kappa \geq \omega$ be a regular cardinal.

- $A, B \in [\kappa]^{\kappa}$ *are said to be* ***almost disjoint*** *or* ***a.d.*** *if* $|A \cap B| < \kappa$.
- *A family $\mathscr{A} \subseteq [\kappa]^{\kappa}$ is said to be* ***almost disjoint*** *or* ***a.d.*** *if the members of $\mathscr{A}$ are pairwise a.d.*
- *Finally $\mathscr{A} \subseteq [\kappa]^{\kappa}$ is called* ***maximal almost disjoint*** *or* ***m.a.d.*** *if $\mathscr{A}$ is an a.d. family, $|\mathscr{A}| \geq \kappa$, and $\mathscr{A}$ cannot be extended to a larger a.d. family in $[\kappa]^{\kappa}$.*

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Definition

$\mathfrak{a}(\kappa) = \min \{|\mathscr{A}| : \mathscr{A} \subseteq [\kappa]^{\kappa}$ *and* $\mathscr{A}$ *is m.a.d.*$\}$.

Theorem (Rothberger)

For any regular $\kappa \geq \omega$, $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$.

Theorem (Shelah)

It is consistent to have $\aleph_1 = \mathfrak{b}(\omega) < \mathfrak{a}(\omega) = \aleph_2 = \mathfrak{s}(\omega)$. *It is also consistent to have* $\aleph_1 = \mathfrak{b}(\omega) = \mathfrak{a}(\omega) < \mathfrak{s}(\omega)$.

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- It turns out that ω is the *only regular* κ where $\mathfrak{b}(\kappa) = \kappa^+ < \kappa^{++} = \mathfrak{a}(\kappa)$ is consistent.

Theorem (R. + Shelah)

If $\kappa > \omega$ *is regular, then* $\mathfrak{b}(\kappa) = \kappa^+$ *implies* $\mathfrak{a}(\kappa) = \kappa^+$.

Theorem (Blass, Hyttinen, and Zhang)

Let $\kappa > \omega$ be regular. If $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.

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Question (Roitman)

Does $\mathfrak{d}(\omega) = \aleph_1$ imply that $\mathfrak{a}(\omega) = \aleph_1$?

Theorem (Shelah)

It is consistent to have $\aleph_2 = \mathfrak{d}(\omega) < \mathfrak{a}(\omega) = \aleph_3$.

- He actually gave two different proofs of $\mathrm{Con}(\mathfrak{d}(\omega) < \mathfrak{a}(\omega))$.
- The first proof used ultrapowers and needed a measurable cardinal θ to produce a model with $\theta < \mathfrak{d}(\omega) < \mathfrak{a}(\omega)$.
- The other proof used templates and produced a model with $\mathfrak{d}(\omega) = \aleph_2$.

- Shelah's first proof also works for $\mathfrak{u}(\omega)$.

Theorem (Shelah)

Suppose there is a measurable cardinal θ. Then there is a c.c.c. forcing extension in which $\theta < \mathfrak{u}(\omega) < \mathfrak{a}(\omega)$.

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Theorem (Shelah)

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Theorem (Guzman and Kalajdzievski)

It is consistent relative to ZFC that $\aleph_1 = \mathfrak{u}(\omega) < \mathfrak{a}(\omega) = \aleph_2$ holds.

- Recall that Suzuki showed that if κ is not weakly compact, then $\mathfrak{s}(\kappa) < \kappa^+$.
- So should it be the case that if κ is not weakly compact, then $\mathfrak{r}(\kappa)$ is "large"?

- Recall that Suzuki showed that if κ is not weakly compact, then $\mathfrak{s}(\kappa) < \kappa^+$.
- So should it be the case that if κ is not weakly compact, then $\mathfrak{r}(\kappa)$ is "large"?

Theorem (R. + Shelah [1])

Suppose that κ is supercompact. There is a forcing extension in which κ becomes the first Mahlo cardinal and $\mathfrak{r}(\kappa) = \kappa^+ < 2^\kappa$.

- If there are no inaccessibles above κ, then 2^κ can be made arbitrary here.

Definition

For an inaccessible cardinal θ define

$$\mathrm{SS}_\theta = \{\mu < \theta : \mu \text{ is a singular strong limit}\}$$

Definition

For an inaccessible θ, we define $\mathbb{Q}_\theta^{\mathrm{am}}$ to be

$$\{p : \exists \alpha < \theta \, [p \text{ is an increasing continuous function from } \alpha \text{ to } \mathrm{SS}_\theta]\}.$$

Definition

Define

$$\mathbb{Q}^{\mathrm{am}}_{<\kappa} = \prod \left\{ \mathbb{Q}^{\mathrm{am}}_{\theta} : \theta < \kappa \textit{ is an inaccessible cardinal} \right\},$$

where the product is taken with Easton support.

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where the product is taken with Easton support.

- By standard arguments, forcing with $\mathbb{Q}^{\mathrm{am}}_{<\kappa}$ will make κ the least Mahlo cardinal.
- Now the idea is to assume κ is Laver indestructible, and first force to make $\mathfrak{r}(\kappa) = \kappa^{+} < 2^{\kappa}$ by a $< \kappa$-directed closed forcing.
- Then do some preparatory forcings which maintain supercompactness.
- Finally forcing with $\mathbb{Q}^{\mathrm{am}}_{<\kappa}$ will make κ the first Mahlo, and the preparatory forcings will have anticipated names, guaranteeing that $\mathfrak{r}(\kappa) = \kappa^{+}$ still holds.

- Suppose κ is Laver indestructible, $\kappa << \mu = \mathrm{cf}(\mu) << \lambda = \lambda^{<\lambda}$ (λ below first inaccessible above κ).
- By the work of Garti and Shelah, and Dzamonja and Shelah, we can force a normal measure $\mathcal{D}$ on κ and a base $\mathcal{A} \subseteq \mathcal{D}$ with $|\mathcal{A}| = \mu$.
- This is only for simplicity. We only need a pseudo-base for a normal measure on κ, which is much easier to achieve.

- We may assume that for each $A \in \mathcal{A}$, $A \subseteq \{\theta < \kappa : \theta \text{ is strongly inaccessible}\}$.

Definition

*For $A \in \mathcal{A}$, a sequence $\bar{p} = \langle p_\theta : \theta \in \mathrm{nacc}(A) \rangle$ is said to be A-**nice** if the following hold:*

1. *each $p_\theta \in \mathbb{Q}^{\mathrm{am}}_{<\kappa}$;*
2. *$\theta \in \mathrm{dom}(p_\theta)$ and there is a fixed $r_{\bar{p}}$ such that $p_\theta(\theta) = r_{\bar{p}}$, for all $\theta \in \mathrm{nacc}(A)$;*
3. *there is a fixed $q_{\bar{p}}$ so that $p_\theta \restriction \theta = q_{\bar{p}}$, for all $\theta \in \mathrm{nacc}(A)$;*
4. *for each $\theta \in \mathrm{nacc}(A_m)$, $\mathrm{dom}(p_\theta) \subseteq \min(A_m \setminus \theta^+)$.*

Note that if $\bar{p}$ is A-nice, then $\mathrm{dom}(\bar{p}) = \mathrm{nacc}(A)$.

Definition

Let $X_{\mathcal{A}}$ be the collection of all $m = \langle A_m, r_m, q_m \rangle$ such that

1. $A_m \in \mathcal{A}$;
2. $\exists \theta < \min(A_m) \left[r_m \in \mathbb{Q}^{\mathrm{am}}_{\theta} \right]$;
3. $\exists \theta < \min(A_m) \, [q_m \in (\mathbb{Q}^{\mathrm{am}}_{<\kappa}) \restriction \theta]$.

For each $m \in X_{\mathcal{A}}$ define

$$N_m = \{\bar{p} : \bar{p} \text{ is } A_m\text{-nice and } r_{\bar{p}} = r_m \text{ and } q_{\bar{p}} = q_m\}.$$

Definition

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For each $m \in X_{\mathcal{A}}$ *define*

$$N_m = \{\bar{p} : \bar{p} \text{ is } A_m\text{-nice and } r_{\bar{p}} = r_m \text{ and } q_{\bar{p}} = q_m\}.$$

- Observe that $X_{\mathcal{A}}$ is not too large – i.e. $|X_{\mathcal{A}}| \leq \mu$.
- If $\bar{p}$ is A-nice, then it determines a $\mathbb{Q}^{\mathrm{am}}_{<\kappa}$-name for a subset of A given by $\mathring{B}_{\bar{p}} = \{\theta \in \mathrm{dom}(\bar{p}) : p_\theta \in \mathring{G}\}$.

- Conversely, if $\mathring{B}$ is a $\mathbb{Q}^{\mathrm{am}}_{<\kappa}$-name for a subset of κ and $p \in \mathbb{Q}^{\mathrm{am}}_{<\kappa}$, then either $p \Vdash$ "$\mathring{B} \equiv 0 \mod \mathcal{D}$" or for some $m \in X_{\mathcal{A}}$ and $\bar{p} \in N_m$, $q_m \leq p$ and

$$q_m \Vdash \text{“}\mathring{B}_{\bar{p}} \in [\kappa]^{\kappa} \text{ and } \mathring{B}_{\bar{p}} \subseteq \mathring{B}\text{”}.$$

- Conversely, if $\mathring{B}$ is a $\mathbb{Q}^{\mathrm{am}}_{<\kappa}$-name for a subset of κ and $p \in \mathbb{Q}^{\mathrm{am}}_{<\kappa}$, then either $p \Vdash$ "$\mathring{B} \equiv 0 \mod \mathcal{D}$" or for some $m \in X_{\mathcal{A}}$ and $\bar{p} \in N_m$, $q_m \leq p$ and

$$q_m \Vdash \text{“}\mathring{B}_{\bar{p}} \in [\kappa]^{\kappa} \text{ and } \mathring{B}_{\bar{p}} \subseteq \mathring{B}\text{”}.$$

Definition

For any $m \in X_{\mathcal{A}}$ define a poset $\mathbb{R}_m$ for adding a generic element of N_m.
$\mathbb{R}_m = N_m \times \kappa$.
For $\langle \bar{p}_1, \gamma_1 \rangle, \langle \bar{p}_2, \gamma_2 \rangle \in \mathbb{R}_m$, $\langle \bar{p}_2, \gamma_2 \rangle \leq \langle \bar{p}_1, \gamma_1 \rangle$ iff

1. $\gamma_2 \geq \gamma_1$;
2. $\bar{p}_1 \restriction \gamma_1 = \bar{p}_2 \restriction \gamma_1$;
3. $\forall \theta \in \mathrm{nacc}(A_m) \, [p_{2,\theta} \leq p_{1,\theta}]$.

Define the $\mathbb{R}_m$-name $\mathring{\bar{p}}_m = \bigcup \left\{ \bar{p} \restriction \gamma : \langle \bar{p}, \gamma \rangle \in \mathring{G} \right\}$.

Definition

Define

$$\mathbb{R}_{\mathcal{A}} = \prod \{\mathbb{R}_m : m \in X_{\mathcal{A}}\},$$

with $< \kappa$ *supports.*

Lemma

Suppose that G *is* $(\mathbf{V}, \mathbb{R}_{\mathcal{A}})$*-generic. For any* $\mathring{B}$*, if* $\mathring{B}$ *is a* $\mathbb{Q}^{\mathrm{am}}_{<\kappa}$*-name for a subset of* κ *and* $p \in \mathbb{Q}^{\mathrm{am}}_{<\kappa}$*, then the following holds in* $\mathbf{V}[G]$*:*

Either $p \Vdash$ "$\mathring{B} \equiv 0 \mod \mathcal{D}$" *or*

for some $m \in X_{\mathcal{A}}, q_m \leq p,$ *and* $q_m \Vdash$ "$\mathring{B}_{\mathring{\bar{p}}_m[G]} \in [\kappa]^{\kappa}$ *and* $\mathring{B}_{\mathring{\bar{p}}_m[G]} \subseteq \mathring{B}$".

- Now we can do a $< \kappa$-support iteration $\langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \mu \rangle$ satisfying the following:
  1. $\mathbb{P}_\mu$ has cardinality λ;
  2. for each $\alpha < \mu$, $\mathbb{P}_\alpha$ preserves the supercompactness of κ;

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  1. $\mathbb{P}_\mu$ has cardinality λ;
  2. for each $\alpha < \mu$, $\mathbb{P}_\alpha$ preserves the supercompactness of κ;
  3. $\mathring{\mathbb{Q}}_0$ adds λ many Cohen subsets to κ;
  4. for even $\alpha < \mu$, $\mathring{\mathbb{Q}}_\alpha$ adds $\mathring{\mathcal{D}}_\alpha$ and $\mathring{\mathcal{A}}_\alpha$ such that $\mathring{\mathcal{D}}_\alpha$ is a name for a normal measure on $\mathcal{P}(\kappa) \cap \mathbf{V}^{\mathbb{P}_\alpha}$, and $\mathring{\mathcal{A}}_\alpha$ is name for a base (or just a pseudo-base) for $\mathring{\mathcal{D}}_\alpha$ having size μ;

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  5. for even $\alpha < \mu$, $\mathring{\mathbb{Q}}_{\alpha+1}$ is a name for $\mathbb{R}_{\mathring{\mathcal{A}}_\alpha}$, adding a sequence $\left\langle \mathring{\bar{p}}_{\alpha,m} : m \in X_{\mathring{\mathcal{A}}_\alpha} \right\rangle$;

- Now we can do a $<\kappa$-support iteration $\langle \mathbb{P}_\alpha ; \mathring{\mathbb{Q}}_\alpha : \alpha \leq \mu \rangle$ satisfying the following:
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  4. for even $\alpha < \mu$, $\mathring{\mathbb{Q}}_\alpha$ adds $\mathring{\mathcal{D}}_\alpha$ and $\mathring{\mathcal{A}}_\alpha$ such that $\mathring{\mathcal{D}}_\alpha$ is a name for a normal measure on $\mathcal{P}(\kappa) \cap \mathbf{V}^{\mathbb{P}_\alpha}$, and $\mathring{\mathcal{A}}_\alpha$ is name for a base (or just a pseudo-base) for $\mathring{\mathcal{D}}_\alpha$ having size μ;
  5. for even $\alpha < \mu$, $\mathring{\mathbb{Q}}_{\alpha+1}$ is a name for $\mathbb{R}_{\mathring{\mathcal{A}}_\alpha}$, adding a sequence $\left\langle \mathring{\bar{p}}_{\alpha, m} : m \in X_{\mathring{\mathcal{A}}_\alpha} \right\rangle$;
- finally in $\mathbf{V}^{\mathbb{P}_\mu \times \mathbb{Q}^{\mathrm{am}}_{<\kappa}}$, we have $\mathfrak{r}(\kappa) \leq \mu$, $2^\kappa = \lambda$, and κ is the first Mahlo cardinal;

- the witness for $\mathfrak{r}(\kappa)$ is just given by

$$\left\{A \in [\kappa]^{\kappa} : \exists \alpha < \mu \left[\alpha \text{ is even and } A \in \mathring{\mathcal{A}}_{\alpha}\left[G_{\mathbb{P}_{\mu}}\right]\right]\right\} \cup$$
$$\left\{\mathring{B}_{\mathring{\bar{p}}_{\alpha,m}[G_{\mathbb{P}_{\mu}}]}\left[G_{\mathbb{Q}^{\mathrm{am}}_{<\kappa}}\right] : \alpha < \mu \text{ and } \alpha \text{ is even and } m \in X_{\mathring{\mathcal{A}}_{\alpha}[G_{\mathbb{P}_{\mu}}]}\right\}.$$

- the witness for $\mathfrak{r}(\kappa)$ is just given by

$$\left\{A \in [\kappa]^{\kappa} : \exists \alpha < \mu \left[\alpha \text{ is even and } A \in \mathring{\mathcal{A}}_{\alpha}\left[G_{\mathbb{P}_{\mu}}\right]\right]\right\} \cup$$
$$\left\{\mathring{B}_{\mathring{\bar{p}}_{\alpha,m}[G_{\mathbb{P}_{\mu}}]}\left[G_{\mathbb{Q}^{\mathrm{am}}_{<\kappa}}\right] : \alpha < \mu \text{ and } \alpha \text{ is even and } m \in X_{\mathring{\mathcal{A}}_{\alpha}[G_{\mathbb{P}_{\mu}}]}\right\}.$$

- The conditions (1)–(5) are all easy to achieve except for (2).
- This is because the forcings $\mathbb{R}_m$ do not satisfy Laver's condition.
- However we are only interested in preserving the supercompactness of κ by forcings of size at most λ, where the interval $[\kappa, \lambda]$ has no strong inaccessibles.

- For this the following condition on a forcing $\mathbb{P}$ suffices:

  for every inaccessible $\theta < \kappa,$ if $\mathcal{P} \subseteq \mathbb{P}$ is a $(< \theta)$-directed set of cardinality less than the next inaccessible above $\theta,$ then $\mathcal{P}$ has a lower bound in $\mathbb{P}$. $(*)$

- It is not hard to show that the forcings $\mathbb{R}_{\mathcal{A}}$ satisfy this condition.

Question

What is the consistency strength of the statement that κ is the first Mahlo cardinal and $\mathfrak{r}(\kappa) = \kappa^+ < 2^\kappa$?

Question

Is it possible to arrange $\mathfrak{r}(\kappa) = \kappa^+ < 2^\kappa$ at the first strongly inaccessible cardinal?

Question

Is is possible to arrange $\mathfrak{u}(\kappa) = \kappa^+ < 2^\kappa$ at the first weakly compact cardinal?

Bibliography

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