The ultrafilter and almost disjointness numbers

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Ultrafilters and MAD families play a fundamental role on infinite combinatorics, set theoretic topology and other branches of mathematics. For this reason, it is interesting to study the relationship between this two objects. We will focus on the *cardinal invariants* associated to each of them.

Definition

Let $\mathcal{U}$ be an ultrafilter and $\mathcal{B} \subseteq \mathcal{U}$. We say that $\mathcal{B}$ is a *base of* $\mathcal{U}$ if for every $A \in \mathcal{U}$, there is $B \in \mathcal{B}$ such that $B \subseteq A$.

1. The *ultrafilter number* $\mathfrak{u}$ denotes the smallest size of a base for an ultrafilter on ω.
2. The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family.

Both $\mathfrak{u}$ and $\mathfrak{a}$ are examples of what we call *cardinal invariants of the continuum.* Both are uncountable cardinals and are at most the cardinality of the real numbers.

$$\omega < \mathfrak{u}, \mathfrak{a} \leq \mathfrak{c}$$

Is there any relationship between $\mathfrak{u}$ and $\mathfrak{a}$?

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1. If we assume the Continuum Hypothesis, then $\omega_1 = \mathfrak{a} = \mathfrak{u} = \mathfrak{c}$.
2. The consistency of the inequality $\mathfrak{a} < \mathfrak{u}$ is well known and easy to prove, in fact, it holds in the Cohen, random and Silver models, among many others.
3. Proving the consistency of the inequality $\mathfrak{u} < \mathfrak{a}$ is much harder and used to be an open problem for a long time. The consistency of this inequality was obtained by Shelah.

There is a reason why the consistency of $\mathfrak{u} < \mathfrak{a}$ must be hard to prove. It follows by the theorems of Džamonja,Hrušák and Moore and that the inequality $\mathfrak{u} < \mathfrak{a}$ can not be obtained by using countable support iteration of proper Borel partial orders.

The consistency of $\mathfrak{u} < \mathfrak{a}$ was finally established by Shelah, when he proved the following theorem:

Theorem (Shelah)

Let V be a model of GCH, κ a measurable cardinal and μ, λ two regular cardinals such that $\kappa < \mu < \lambda$. There is a c.c.c. forcing extension of V that satisfies $\mu = \mathfrak{b} = \mathfrak{d} = \mathfrak{u}$ and $\lambda = \mathfrak{a} = \mathfrak{c}$. In particular, CON(ZFC + "there is a measurable cardinal") implies CON(ZFC + "$\mathfrak{u} < \mathfrak{a}$").

Theorem (Shelah)

Let V be a model of GCH, κ a measurable cardinal. There is a c.c.c. forcing extension of V that satisfies $\mathfrak{u} = \kappa^+$ and $\mathfrak{a} = \mathfrak{c} = \kappa^{++}$. In particular, CON(ZFC + "there is a measurable cardinal") implies CON(ZFC + "$\mathfrak{u} < \mathfrak{a}$").

This theorem was one of the first results proved using "template iterations", which is a very powerful method that has been very useful and has been successfully applied to this day. In spite of the beauty of this result, it leaves open the following questions:

Problem (Shelah)

Does CON(ZFC) imply CON(ZFC + "$\mathfrak{u} < \mathfrak{a}$")?

Problem (Brendle)

Is it consistent that $\omega_1 = \mathfrak{u} < \mathfrak{a}$?

With Damjan Kalajdzievski, we were able provide a positive answer to both questions, by proving (without appealing to large cardinals) that every MAD family can be destroyed by a proper forcing that preserves P-points.

Definition

We say that $\mathcal{I} \subseteq \wp(\omega)$ is an *ideal* if the following conditions hold:

1. $[\omega]^{<\omega} \subseteq \mathcal{I}$.
2. If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.
3. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
4. $\omega \notin \mathcal{I}$.

We say that $\mathcal{I}$ is *tall* if for every $X \in [\omega]^{\omega}$, there is $A \in \mathcal{I}$ such that $A \cap X$ is infinite.

Destroying ideals with forcing

Let $\mathcal{I}$ be an ideal on ω, $\mathcal{F}$ a filter on ω, $\mathcal{U}$ an ultrafilter on ω and $\mathcal{A}$ a MAD family.

1. We say that a forcing notion $\mathbb{P}$ *destroys* $\mathcal{I}$ if $\mathbb{P}$ adds an infinite subset of ω that is almost disjoint with every element of $\mathcal{I}$. In other words, $\mathcal{I}$ is no longer tall after forcing with $\mathbb{P}$.
2. We say that $\mathbb{P}$ *diagonalizes* $\mathcal{F}$ if $\mathbb{P}$ adds an infinite set almost contained in every element of $\mathcal{F}$.
3. We say that $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$.
4. We say that $\mathbb{P}$ *preserves* $\mathcal{U}$ if after forcing with $\mathbb{P}$, $\mathcal{U}$ is the base of an ultrafilter.

Some simple remarks:

1. $\mathbb{P}$ destroys the ideal $\mathcal{I}$ if and only if $\mathbb{P}$ diagonalizes the filter $\mathcal{I}^* = \{\omega \setminus A \mid A \in \mathcal{I}\}$.
2. $\mathbb{P}$ destroys $\mathcal{A}$ if and only if $\mathbb{P}$ destroys the ideal generated by $\mathcal{A}$ (denoted by $\mathcal{I}(\mathcal{A})$).

1. Let $f, g \in \omega^{\omega}$, define $f \leq^{*} g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^{\omega}$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^{*}$.
2. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family.
3. We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is a *splitting family* if for every $X \in [\omega]^{\omega}$ there is $S \in \mathcal{S}$ such that S splits X.
4. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family.

Our motivation comes from the theorems of Shelah that establishes that the statements "$\omega_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \omega_2$" and "$\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \omega_2$" are consistent. After this results, different models of $\omega_1 = \mathfrak{b} < \mathfrak{a} = \omega_2$ were constructed by Dow, Brendle, Steprāns and Fischer, among others. In every case, the forcings used add Cohen reals, so no ultrafilter is preserved.

In order to construct the models of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$, Shelah used a creature forcing. Later, Brendle and Raghavan found a simpler representation of his forcing as a two step iteration, which we will briefly describe.

The most natural way to increase the splitting number is to diagonalize an ultrafilter. This is because diagonalizing an ultrafilter destroys all splitting families of the ground model. In order to build a model of $\mathfrak{b} < \mathfrak{s}$, it is enough to construct (or force) an ultrafilter that can be diagonalized without adding dominating reals (even in the iteration).

Denote by $\mathbb{F}_\sigma$ the set of all F_σ-filters on ω. If $\mathcal{F}, \mathcal{G} \in \mathbb{F}_\sigma$ we define $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{F}$. It is not hard to see that $\mathbb{F}_\sigma$ naturally adds an ultrafilter $\dot{\mathcal{U}}_{gen}$. This forcing was introduced by Laflamme, has also been studied by Mildenberger.

It was proved by Brendle and Raghavan that the forcing of Shelah is equivalent to the two step iteration $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$, where $\mathbb{M}(\mathcal{U}_{gen})$ is the Mathias forcing relative to $\mathcal{U}_{gen}$. It can be proved that $\mathbb{M}(\mathcal{U}_{gen})$ does not add dominating reals, even when iterated.

The method to build a model of $\mathfrak{b} < \mathfrak{a}$ is similar: Given a MAD family $\mathcal{A}$, denote by $\mathbb{F}_\sigma(\mathcal{A})$ the set of all F_σ-filters $\mathcal{F}$ such that $\mathcal{F} \cap \mathcal{I}(\mathcal{A}) = \emptyset$ (where $\mathcal{I}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$). Once again, we order $\mathbb{F}_\sigma(\mathcal{A})$ with inclusion. It is easy to see that $\mathbb{F}_\sigma(\mathcal{A})$ naturally adds an ultrafilter $\mathcal{U}_{gen}(\mathcal{A})$, furthermore, diagonalizing $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ destroys the maximality of $\mathcal{A}$.

Brendle and Raghavan proved that $\mathbb{C}_{\omega_1} * \mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ ($\mathbb{C}_{\omega_1}$ denotes the forcing for adding ω_1-Cohen reals) is equivalent to the creature forcing of Shelah. It does not add dominating reals, even when iterated. We want to point out that in the original applications, it was required to add Cohen reals in an explicit way. Our work shows that adding the Cohen reals was in fact not needed.

How could we build a model of $\mathfrak{u} < \mathfrak{s}$?

A strategy to construct such model, would be to diagonalize an ultrafilter while preserving another ultrafilter (in fact, we need to preserve a P-point).

We could try to force with $\mathbb{F}_\sigma$ and then diagonalize $\dot{\mathcal{U}}_{gen}$ in a "nice" way (while preserving a P-point).

How could we build a model of $\mathfrak{u} < \mathfrak{a}$?

Given a MAD family $\mathcal{A}$, we can force with $\mathbb{F}_\sigma(\mathcal{A})$ and then diagonalize the generic ultrafilter in a “nice” way (while preserving a P-point).

Diagonalizing filters

There are two well known forcings used for diagonalizing a filter $\mathcal{F}$:

Definition

The *Laver forcing* $\mathbb{L}(\mathcal{F})$ *with respect to* $\mathcal{F}$ is the set of all trees p such that $suc_p(s) \in \mathcal{F}$ for every $s \in p$ extending the stem of p (where $suc_p(s) = \{n \mid s^\frown n \in p\}$). We say $p \leq q$ if $p \subseteq q$.

Definition

If $\mathcal{F}$ is a filter on ω (or on any countable set) we define the *Mathias forcing* $\mathbb{M}(\mathcal{F})$ *with respect to* $\mathcal{F}$ as the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{F}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{F})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

1. t is an initial segment of s.
2. $A \subseteq B$.
3. $(s \setminus t) \subseteq B$.

Both $\mathbb{L}(\mathcal{F})$ and $\mathbb{M}(\mathcal{F})$ diagonalize $\mathcal{F}$, however, we have the following:

Lemma

Let $\mathcal{F}$ be a filter:

1. $\mathbb{L}(\mathcal{F})$ *adds a dominating real.*
2. $\mathbb{M}(\mathcal{F})$ *adds a Cohen real if and only if $\mathcal{F}$ is not a Ramsey ultrafilter.*
3. *If $\mathcal{F}$ is a Ramsey ultrafilter, then $\mathbb{M}(\mathcal{F})$ adds a dominating real.*
4. *In particular, $\mathbb{M}(\mathcal{F})$ adds either a dominating real or a Cohen real.*

It is known that adding a dominating real or a Cohen real will destroy all ultrafilters.

In this way, the Laver and Mathias forcings (based on a filter) does not help us for our problem. We need a different way to diagonalize filters.

Let $p \subseteq \omega^{<\omega}$ be a tree. If $s \in p$, define $suc_p(s) = \{n \mid s^\frown n \in p\}$. In this talk, we will say that $s \in p$ is a *splitting node* if $suc_p(s)$ is **infinite.**

Definition

We say that a tree $p \subseteq \omega^{<\omega}$ is a *Miller tree* ($p \in \mathbb{PT}$) if the following conditions hold:

1. p consists of increasing sequences.
2. p has a stem (t is the stem of p if every node of p is compatible with t and t is maximal with this property).
3. For every $s \in p$, there is $t \in p$ such that $s \subseteq t$ and t is a splitting node.

We do not require that every node is a splitting node or has only one inmediate succesor.

Let $\mathcal{F}$ be a filter in ω. We say that $X \in \mathcal{F}^+$ if $A \cap X$ is infinite for every $A \in \mathcal{F}$.

Sabok and Zapletal introduced the following parametrized version of Miller forcing:

Definition

Let $\mathcal{F}$ be a filter. By $\mathbb{Q}(\mathcal{F})$ we denote the set of all Miller trees $p \in \mathbb{PT}$ such that $suc_p(s) \in \mathcal{F}^+$ for every splitting node s. The order of $\mathbb{Q}(\mathcal{F})$ is inclusion.

This is a very interesting forcing. Unfortunately, $\mathbb{Q}(\mathcal{F})$ may not diagonalize $\mathcal{F}$.

Lemma

1. $\mathbb{Q}((fin \times fin)^*)$ *does not destroy* $fin \times fin$.
2. $\mathbb{Q}(nwd^*)$ *destroys nwd.*

Definition

Let $\mathcal{F}$ be a filter on ω. Define the filter $\mathcal{F}^{<\omega}$ in $[\omega]^{<\omega} \setminus \{\emptyset\}$ as the filter generated by $\left\{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\right\}$.

Note that if $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in (\mathcal{F}^{<\omega})^{+}$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$.

If $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in \left(\mathcal{F}^{<\omega}\right)^+$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$.

By $split(p)$ we denote the collection of all splitting nodes and by $split_n(p)$ we denote the collection of n-splitting nodes (i.e. $s \in split_n(p)$ if $s \in split(p)$ and s has exactly n-restrictions that are splitting nodes). Given $p \in \mathbb{PT}$ for every $s \in split_n(p)$ we define $F(p, s) = \{t \setminus s \mid t \in split_{n+1}(p) \wedge s \subseteq t\}$.

Definition

Let $\mathcal{F}$ be a filter. We say $p \in \mathbb{PT}(\mathcal{F})$ if the following holds:

1. $p \in \mathbb{PT}$.
2. If $s \in split(p)$ then $F(p, s) \in \left(\mathcal{F}^{<\omega}\right)^+$.

We order $\mathbb{PT}(\mathcal{F})$ by inclusion.

It can be proved that $\mathbb{PT}(\mathcal{F})$ is proper, diagonalizes $\mathcal{F}$ and in some cases, it might preserve some ultrafilters.

Definition

Let $\mathcal{I}$ be an ideal on ω. We define $\mathbb{F}_\sigma(\mathcal{I})$ as the collection of all F_σ-filters $\mathcal{F}$ such that $\mathcal{F} \cap \mathcal{I} = \emptyset$. We order $\mathbb{F}_\sigma(\mathcal{I})$ by inclusion.

Lemma

Let $\mathcal{I}$ be an ideal on ω.

1. $\mathbb{F}_\sigma(\mathcal{I})$ *is a σ-closed forcing.*
2. $\mathbb{F}_\sigma(\mathcal{I})$ *adds an ultrafilter (which we will denote by $\mathcal{U}_{gen}(\mathcal{I})$) disjoint from $\mathcal{I}$.*
3. $\mathbb{F}_\sigma(\mathcal{I}) * \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{I}))$ *and $\mathbb{F}_\sigma(\mathcal{I}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{I}))$ are proper forcings that destroy $\mathcal{I}$.*

If $\mathcal{A}$ is a MAD family, we will denote $\mathbb{F}_\sigma(\mathcal{A})$ instead of $\mathbb{F}_\sigma(\mathcal{I}(\mathcal{A}))$ and $\mathcal{U}_{gen}(\mathcal{A})$ instead of $\mathcal{U}_{gen}(\mathcal{I}(\mathcal{A}))$. Note that $\mathbb{F}_\sigma\left([\omega]^{<\omega}\right)$ is the collection of all F_σ-filters. In this case, we will only denote it by $\mathbb{F}_\sigma$ and by $\mathcal{U}_{gen}$ we will denote the generic ultrafilter added by $\mathbb{F}_\sigma$.

The bounding and splitting numbers

Brendle and Raghavan proved that the creature forcing used by Shelah to get $\mathfrak{b} < \mathfrak{s}$ is forcing equivalent to $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$. We know iterating this forcing with countable support yields a model of $\omega_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{u} = \mathfrak{s}$. We proved that iterating $\mathbb{F}_\sigma * \mathbb{PT}(\dot{\mathcal{U}}_{gen})$ gives a model of $\omega_1 = \mathfrak{b} = \mathfrak{a} = \mathfrak{u} < \mathfrak{s}$.

The consistency of this inequality is not new, it holds in the Shelah-Blass model.

The bounding and almost disjointness numbers

Brendle and Raghavan proved that the creature forcing used by Shelah to get $\mathfrak{b} < \mathfrak{a}$ is forcing equivalent to $\mathbb{C}_{\omega_1} * \mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$. In the original arguments, it was necessary to add Cohen reals as a preliminary step. Fortunately, this step can be avoided. An iteration of the forcings $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ gives a model of $\omega_1 = \mathfrak{b} < \mathfrak{a} = \mathfrak{u} = \mathfrak{s}$. On the other hand, it can be proved that iterating $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ gives a model of $\omega_1 = \mathfrak{b} = \mathfrak{u} < \mathfrak{a} = \mathfrak{s}$.

While $\mathbb{L}(\mathcal{F})$ always adds a dominating real, this may not be the case for $\mathbb{M}(\mathcal{F})$. A trivial example is taking $\mathcal{F}$ to be the cofinite filter in ω, since in this case $\mathbb{M}(\mathcal{F})$ is forcing equivalent to Cohen forcing. A more interesting example was found by Canjar, where an ultrafilter whose Mathias forcing does not add dominating reals was constructed under $\mathfrak{d} = \mathfrak{c}$.

Definition

We say that a filter $\mathcal{F}$ is *Canjar* if $\mathbb{M}(\mathcal{F})$ does not add dominating reals.

Theorem

Let $\mathcal{F}$ be a filter on ω. The following are equivalent:

1. *$\mathcal{F}$ is Canjar.*
2. *(Hrušák, Minami) For every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$.*
3. *(Chodounský, Repovš and Zdomskyy) $\mathcal{F}$ is Menger (as a subspace of $\wp(\omega) \simeq 2^\omega$).*

Theorem

1. *(Brendle) If $\mathcal{F}$ is F_σ, then $\mathcal{F}$ is Canjar.*
2. *(G., Martínez-Celis, Hrušák and Chodounský, Repovš and Zdomskyy independently) If $\mathcal{F}$ is a Canjar Borel filter, then $\mathcal{F}$ is F_σ.*
3. *(Laflamme) $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is Canjar.*
4. *(Shelah + Brendle and Raghavan) Let $\mathcal{A}$ be a MAD family. $\mathbb{C}_{\omega_1} * \mathbb{F}_\sigma(\mathcal{A})$ forces that $\dot{\mathcal{U}}_{gen}(\mathcal{A})$ is Canjar.*

The Miller forcing of a filter seems to be "nicer" than its Mathias counterpart:

Theorem

If $\mathcal{F}$ is Canjar, then $\mathbb{PT}(\mathcal{F})$ does not add dominating reals.

In this way, if $\mathcal{F}$ is Canjar, then $\mathbb{M}(\mathcal{F})$ and $\mathbb{PT}(\mathcal{F})$ do not add dominating reals. However, $\mathbb{M}(\mathcal{F})$ will add Cohen reals, while $\mathbb{PT}(\mathcal{F})$ might even preserve some ultrafilter.

It might be tempting to conjecture the following: "If $\mathcal{F}$ is Canjar, then $\mathbb{PT}(\mathcal{F})$ preserves P-points."

However, this is false. If $\mathcal{U}$ is a Canjar P-point, then $\mathbb{PT}(\mathcal{U})$ diagonalizes $\mathcal{U}$, so it does not preserve $\mathcal{U}$. Nevertheless, we have the following:

Theorem

Let $\mathcal{W}$ be a P-point and $\mathcal{A}$ a MAD family.

1. *If $\mathcal{F}$ is an F_σ-filter, then $\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{W}$.*
2. $\mathbb{F}_\sigma$ *forces that* $\mathbb{PT}(\dot{\mathcal{U}}_{gen})$ *preserves* $\mathcal{W}$.
3. $\mathbb{F}_\sigma(\mathcal{A})$ *forces that* $\mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ *preserves* $\mathcal{W}$.

In this way, $\mathbb{PT}\left(\dot{\mathcal{U}}_{gen}\right)$ (and $\mathbb{PT}\left(\dot{\mathcal{U}}_{gen}\left(\mathcal{A}\right)\right)$) destroy P-points, but they only destroy "new" P-points, all of the ground model P-points will survive.

In this way, by iterating forcings of the type $\mathbb{F}_{\sigma}\left(\mathcal{A}\right) * \mathbb{PT}\left(\dot{\mathcal{U}}_{gen}\left(\mathcal{A}\right)\right)$ over a model of the Continuum Hypothesis, we will obtain a model of $\omega_1 = \mathfrak{u} < \mathfrak{a}$.