Ideal topologies on 2^κ

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- These topologies are generated by ideals
- The setting is connected with cardinal characteristics for 2^κ
- This is joint work with Peter Holy, Marlene Koelbing and Wolfgang Wohofsky

Ideal topologies

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The *bounded topology* on 2^κ has the basic clopen sets

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where $t \in 2^{<\kappa}$.

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Let κ be a regular and uncountable cardinal. Let NS_κ denote the non-stationary ideal on κ. In a nutshell, the I-topology is obtained from the bounded topology by working with a $<\kappa$-closed ideal containing NS_κ.

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Definition

Let $\mathsf{Fun}_I = \{f : A \to 2 \mid A \in I\}$. The *$I$-topology* on 2^κ has the basic clopen sets

$$[f] = \{g \in 2^\kappa \mid f \subseteq g\},$$

where $f \in \mathsf{Fun}_I$.

Some, but not all of our results also apply to the generalized Baire space κ^κ rather than 2^κ.

Some basic observations

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The intersection of κ many I-open dense sets is I-dense.

On the other hand, the I-topology cannot be characterized by converging sequences.

An ideal on κ is called *tall* if every unbounded subset of κ contains an unbounded subset in I. For instance, any ideal $I \supseteq \mathrm{NS}_\kappa$ is tall.

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Proposition

Assume that I is tall. Then every I-convergent sequence (of any length) is eventually constant.

How does the hierarchy of I-Borel sets look like?

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Proposition

If $T \subseteq \mathsf{Fun}_I$, then

$$[T] = \{x \in 2^\kappa \mid x \| I \subseteq \mathsf{Fun}_I\}$$

is an I-closed subset of 2^κ. Conversely, every I-closed subset of 2^κ is of the form $[T]$ for some $T \subseteq \mathsf{Fun}_I$ that is closed under restrictions.

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Proof.

If $X \subseteq 2^\kappa$ is I-closed, let

$$T = \{x \| A \mid x \in X \wedge A \in \mathrm{NS}_\kappa\}.$$

If $x \in X$, then clearly $x \| I \subseteq T$. Now take $x \notin X$. Since X is I-closed, there is $A \in I$ with $X \cap [x \upharpoonright A] = \varnothing$. But then $x \upharpoonright A \notin T$, hence also $x \| I \nsubseteq T$.

The first statement of the proposition is verified similarly. □

An I-open set that is not I-F_σ

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Proposition

U is not I-F_σ, i.e. no κ-union of I-closed sets.

Proof.

Assume for a contradiction that it is, i.e. $U = \bigcup_{\alpha<\kappa}[T_\alpha]$, with each $T_\alpha \subseteq \mathsf{Fun}_I$. We inductively construct an unbounded subset of κ which is not in U. We say that $f \in \mathsf{Fun}_I$ is *bounded* in κ if $\{\gamma < \kappa \mid f(\gamma) = 1\}$ is.

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Starting with $f_0 = \varnothing$, we construct a continuous and increasing κ-sequence of bounded f_α's so that $f_{\alpha+1}(\gamma) = 1$ for some $\gamma \geqslant \alpha$, and so that $f_{\alpha+1} \notin T_\alpha$ for all $\alpha < \kappa$: If some T_α contained all bounded extensions of f_α, then $[T_\alpha]$ would have to contain a bounded set. In the end, $f = \bigcup_{\alpha<\kappa} f_\alpha$ is an unbounded subset of κ which is not in U, yielding our desired contradiction. □

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One can show similarly that the set of clubs in κ is not I-F_σ, for $I = \mathrm{NS}_\kappa$.

The club filter is not I-Borel

Let $I = \mathrm{NS}_\kappa$. Note first that the club filter is both I-dense and co-dense. Similar to the Baire category theorem, one can show that every κ-intersection of I-open dense sets contains both an element of the club filter, and of the nonstationary ideal.

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By a similar argument as for the bounded topology, the club filter cannot have the I-Baire property.

Lemma

For $I = \mathrm{NS}_\kappa$, the club filter doesn't have the I-Baire property. In particular, it's not I-Borel.

What's the relation between I-meager and meager?

I-meager sets

From now on, let $I = \mathrm{NS}_\kappa$. Recall:

Definition

- A subset A of 2^κ is *I-nowhere dense* if for each $f \in \mathsf{Fun}_I$, there's some $g \in \mathsf{Fun}_I$ with $f \subseteq g$ and $[g] \cap A = \varnothing$.
- A is *I-meager* if it is a κ-union of I-nowhere dense sets.
- A has the *I-Baire property* if it is of the form $U \triangle M$, where U is I-open and M is I-meager.

We call the sets $[f]$ *I-cones*. By the Baire category theorem, these are not I-meager.

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Basic properties of I-nowhere dense sets:

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Basic properties of I-nowhere dense sets:

- every set of size $< 2^\kappa$ is I-nowhere dense
- there is an I-nowhere dense set of size 2^κ:

$$\{x \in 2^\kappa \mid x(\alpha) = x(\alpha+1) \text{ for each even } \alpha < \kappa\}.$$

I-meager versus meager

If $f \in \mathsf{Fun}_I$ and $|\mathsf{dom}(f)| = \kappa$, then $[f]$ is closed nowhere dense. Hence:

Proposition

There is a meager set which is not I-meager.

I-meager versus meager

The converse direction is more subtle.

Lemma

Assume κ is inaccessible or $\diamondsuit_\kappa$ holds.

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Assume κ is inaccessible or $\diamondsuit_\kappa$ holds.
Then every comeager set contains an I-cone $[f]$:

$$\text{For } \vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle \text{ open dense} \quad \exists f \in \mathsf{Fun}_I \quad [f] \subseteq \bigcap_{\alpha<\kappa} D_\alpha .$$

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The assumption holds for all successor cardinals $\kappa = \lambda^+$ with $\lambda > \omega$ and $2^\lambda = \lambda^+$ by a result of Shelah from 2007.

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Lemma

Assume κ *is inaccessible or* $\diamondsuit_\kappa$ *holds. For any* $\vec{D} = \langle D_i \mid i < \kappa \rangle$ *open dense*
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Theorem

Assume that $I \supseteq \mathrm{NS}_\kappa$ *and the conclusion of the lemma holds. If* A *has the Baire property, then "*A *is* I*-meager" implies "*A *is meager".*

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Proof (Theorem).

Assume that A has the Baire property and is not meager. We show that A is not I-meager.

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Since A has the Baire property, there is an $s \in \mathsf{Fun}_{\mathrm{bd}_\kappa}$ such that $A \cap [s]$ is comeager in $[s]$, i.e. there is $\vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle$ open dense, with $\bigcap_{\alpha<\kappa} D_\alpha \cap [s] \subseteq A$.

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By our assumption, there exists $f \supseteq s$, $f \in \mathsf{Fun}_I$ with $[f] \subseteq \bigcap_{\alpha<\kappa} D_\alpha \cap [s] \subseteq A$. Thus A is not I-meager. □

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Proof sketch, part 1.

Fix a $\diamondsuit_\kappa$-sequence $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$, that is, for every $A \subseteq \kappa$, there is a stationary set of α's with $A_\alpha = A \cap \alpha$.

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Assume that $\vec{D}$ is decreasing. By induction on $i < \kappa$, we define

- a continuous $\subseteq$-increasing sequence $\vec{f} = \langle f_i \mid i < \kappa \rangle$ of functions in $\mathsf{Fun}_{\mathrm{bd}_\kappa}$, such that $[f_{i+1}] \subseteq D_i$ for every $i < \kappa$, and
- a club subset $C = \{\alpha_j \mid j < \kappa\}$ of κ that is disjoint from $\mathsf{dom}(f_i)$ for each $i < \kappa$.

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- a club subset $C = \{\alpha_j \mid j < \kappa\}$ of κ that is disjoint from $\mathsf{dom}(f_i)$ for each $i < \kappa$.

Let $f_0 = s$, and pick $\alpha_0 > \sup(\mathsf{dom}(s))$. □

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- h_i^0 extends f_i,
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Now pick $h_i^1 \in \mathsf{Fun}_{\mathrm{bd}_\kappa}$ such that

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- $[h_i^1] \subseteq D_i$.

Let $f_{i+1} = h_i^1 \restriction (\mathsf{dom}(h_i^1) \setminus \{\alpha_j \mid j \leqslant i\})$, and pick some $\alpha_{i+1} > \sup(\mathsf{dom}(f_{i+1}))$.

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Given $x \in [f]$, let $A = \{i < \kappa \mid x(\alpha_i) = 1\}$. Let $i < \kappa$ with $A \cap i = A_i$ by $\diamondsuit_\kappa$.

By the construction of f_{i+1}, we have $x \in [h_i^0] \subseteq D_i$ or $x \in [h_i^1] \subseteq D_i$. So x is in the intersection of the D_i, as desired. □

I-meager versus I-nowhere dense

A similar argument shows the following:

Lemma

Let $I = \mathrm{NS}_\kappa$. Assume that κ is inaccessible or $\diamondsuit_\kappa$ holds. Then for every $f \in \mathsf{Fun}_I$, every κ-intersection of I-open dense sets contains an I-cone $[g]$ with $f \subseteq g$.

Theorem

Assume that κ is inaccessible or $\diamondsuit_\kappa$ holds. Then every I-meager set is I-nowhere dense.

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Theorem

Assume that κ is inaccessible or $\diamondsuit_\kappa$ holds. Then every I-meager set is I-nowhere dense.

Proof.

Suppose that A is disjoint from $U = \bigcap_{i<\kappa} U_i$, where each U_i is I-open dense. Now take any I-cone $[f]$. By the lemma, we can find an I-cone $[g] \subseteq [f]$ disjoint from U. Hence A is not dense in $[f]$. □

I-meager versus meager

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Definition

A reaping family on κ is a set $\mathcal{R} \subseteq [\kappa]^\kappa$ such that

no $a \in [\kappa]^\kappa$ splits all $y \in \mathcal{R}$.

$\mathfrak{r}(\kappa)$ is the smallest size of a reaping family on κ.

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Theorem (κ inaccessible)

Assume $\mathfrak{r}(\kappa) = 2^\kappa$.
Then there is an I-nowhere dense set which is not meager.

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Let $\mathsf{Fun}_{\mathrm{ub}_\kappa}$ denote the set of partial functions from κ to 2 with $|\mathsf{dom}(f)| = \kappa$.

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Definition

R is the smallest size of a family $\mathcal{F} \subseteq \mathsf{Fun}_{\mathrm{ub}_\kappa}$ such that $\bigcup_{f \in \mathcal{F}} [f] = 2^\kappa$. (Call this a *cone covering family*.)

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Lemma ($|2^{<\kappa}| = \kappa$)

$R = \mathfrak{r}$.

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Lemma (κ inaccessible)

R is the smallest size of a family $\mathcal{F} \subseteq \mathsf{Fun}_{\mathrm{ub}_\kappa}$ such that $\bigcup_{f \in \mathcal{F}} [f]$ is comeager.

Lemma ($|2^{<\kappa}| \leqslant \kappa^+$)

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$R(\kappa) \leqslant \mathfrak{r}^*(\kappa)$: Let $\mathcal{F}$ be a strong reaping family at κ. Let c_x^A denote the function with domain A and constant value x.

Then $\{c_i^b \mid b \in \mathcal{F},\, i \in 2\}$ is a cone covering family for 2^κ: For every $x \in 2^\kappa$, there is $b \in \mathcal{F}$ and $i \in 2$ such that $x^{-1}(i) \cap b = \varnothing$. So $x \in [c_{1-i}^b]$.

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$\mathfrak{r}^*(\kappa) \leqslant R(\kappa)$: Let $\mathcal{C}$ be a cone covering family at κ. Let

$$\mathcal{F} := \{f^{-1}(i) \mid f \in \mathcal{C}, i \in 2\} \cap \mathrm{ub}_\kappa.$$

For any $a \subseteq \kappa$, there is $f \in \mathcal{C}$ with $\chi_a \in [f]$. Then $f^{-1}(\{0\}) \cap a = \varnothing$ and $f^{-1}(\{1\}) \cap (\kappa \backslash a) = \varnothing$.

Since $\mathsf{dom}(f) \in \mathrm{ub}_\kappa$, $f^{-1}(\{0\})$ or $f^{-1}(\{1\})$ is unbounded and hence in $\mathcal{F}$. □

I-meager versus meager

Remark

Assume that κ is inaccessible. If $\text{non}(\mathcal{M}) < 2^\kappa$ or $\mathfrak{r}(\kappa) = 2^\kappa$, then we've seen that there's a non-meager set which is I-nowhere dense.

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The missing case is $\mathfrak{r}(\kappa) < \mathrm{non}(\mathcal{M}) = 2^\kappa$. It's open whether this configuration is consistent:

- $\mathfrak{r}(\kappa) < 2^\kappa$ is consistent for various κ (see Dilip's talk).
- $\mathfrak{b}(\kappa) \leqslant \mathfrak{r}(\kappa)$ holds for all regular κ. Moreover by Raghavan and Shelah (2018): $\mathfrak{d}(\kappa) \leqslant \mathfrak{r}(\kappa)$ for regular $\kappa \geqslant \beth_\omega$.
- Brendle, Brooke-Taylor, Friedman and Montoya (2016) ask whether

$$\mathfrak{b}(\kappa) < \mathrm{non}(\mathcal{M})$$

  is consistent for inaccessibles. This seems to be open (and possibly harder) for successor cardinals κ with $\kappa^{<\kappa} = \kappa$ as well.

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Is it consistent that $\mathfrak{r}(\kappa) < \mathrm{non}(\mathcal{M}) = 2^\kappa$?

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Is it consistent that the covering number of I-meager sets is $< 2^\kappa$?

Literature

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Thank you!