Algebraic and Geometric Categorification

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December 1–6, 2019

1 Overview of the field

The idea of categorification is a broad framework that brings new tools to classical mathematical and physical problems, while at the same time providing a unifying perspective tying together seemingly disparate fields. The general philosophy is to view certain mathematical structures as shadows of deeper ones.

For instance, it is a well-established idea that it can be useful to view a positive integer or polynomial as the dimension or graded dimension of a vector space. Algebraic operations such as multiplication of integers lifts to tensor product of vector spaces; however, the morphisms between vector spaces add an extra layer of richness the integers do not see. In turn, it may be possible to realize this vector space as the (complexified) Grothendieck group of a category, thereby explaining the presence of structures such as canonical bases with good positivity properties. Additional symmetry such as a group action on the underlying vector space can be lifted to an action of a monoidal category or 2-category, leading to powerful new tools to study the original simpler objects.

The origin of this point of view goes back to the classical work of Kazhdan-Lusztig [31] on Hecke algebras and the related Kazhdan-Lusztig conjecture for character formulae in the Bernstein-Bernstein-Gelfand category $\mathcal{O}$ proved in [3, 11], as well as the work of Bernstein and Zelevinsky on representation theory of affine Hecke algebras [6], and subsequent work of Lusztig [38], Kashiwara [29] and Nakajima [41] on canonical/crystal bases for quantum groups.

Some of the next breakthroughs, roughly 20 years ago now, include work of Ariki [2], Grojnowski [24], Chuang-Rouquier [17] and others on symmetric groups, and Khovanov's categorification of the Jones polynomial [32], which has led to a new industry in knot theory and low-dimensional topology. These and other important results have allowed categorification to emerge as an important subfield of mathematics, combining aspects of representation theory, combinatorics, algebraic geometry, geometric topology, and even mathematical physics.

Important active approaches to categorification include the following.

- *Diagrammatic categorification:* The diagrammatic approach to categorification was pioneered by Khovanov and Lauda [35] in their development of the theory of Kac-Moody 2-categories, which were discovered independently by Rouquier [45]. The diagrammatic point of view has seen dramatic success in the work of Elias and Williamson [21] on the positivity of Kazhdan-Lusztig polynomials via Hecke categories and the Soergel bimodules introduced originally in [48]. More recently, Makisumi and others [1] have used the Soergel calculus developed by Elias and Williamson to establish Koszul duality theorems in a general algebraic context, extending classical results of Beilinson, Ginzburg and Soergel [4] on the BGG category $\mathcal{O}$, while Elias and Hogancamp [19] have introduced an important new technique of categorical diagonalization.

- *Geometric categorification:* The Hecke categories and Kac-Moody 2-categories just mentioned have their roots in the classic geometric work of Kazhdan and Lusztig on Hecke algebras, and the subsequent work of Lusztig, Nakajima and others on quantum groups mentioned earlier. In the last few years, a geometric approach to categorification, using categories of coherent sheaves, D-modules and DQ-modules, has been extensively developed by Cautis and Kamnitzer [12, 13] in the context of the affine Grassmannian, and Losev, Li, Webster and others in the context of quiver varieties [7, 37, 50, 49]. In addition, categorification of the Heisenberg algebra has been used to study the geometry of Hilbert schemes of points on surfaces [16]. Forthcoming work of Bezrukavnikov and Okounkov connects this perspective to enumerative geometry: specifically, they show that wall-crossing functors on the Grothendieck group of coherent sheaves matches the quantum connection, whose connections to K-theory have already been extensively studied by Okounkov and others [40, 42].

- *Combinatorial categorification:* The combinatorial approach to categorification exploits the theory of crystal bases, and cells in Weyl groups. These sorts of rich combinatorial structures often underpin categorifications, and have produced many important applications. Modular branching rules for symmetric groups and, more generally, cyclotomic Hecke algebras, have been related to crystal graphs for representations of affine Lie algebras [36, 47]. In addition, recent work of Dudas, Varagnolo, and Vasserot has exploited crystals arising from unipotent representations of finite classical groups to understand modular Harish-Chandra series [18].

All of the above-mentioned areas have promising, but not deeply explored, relations to mathematical physics. In particular, knot homology, geometric Langlands, Koszul duality and many other topics discussed above have been interpreted by physicists in terms of dualities between field theories, beginning with celebrated work of Kapustin and Witten [28, 22].

2 Some recent developments

2.1 Soergel bimodules and the Lusztig conjecture

One of the most exciting growth areas in the last few years has been the new approach to the theory of Soergel bimodules via generators and relations which are encapsulated in the diagrammatic Soergel calculus developed by Elias and Williamson. This has led to some very remarkable progress in representation theory, algebraic geometry, knot theory, and algebraic combinatorics. In particular, this work has given us a purely algebraic proof (avoiding perverse sheaves or D-modules) of the Kazhdan–Lusztig conjectures on the multiplicities of simple modules in Verma modules for complex semisimple Lie algebra [21], and proof of the positivity, monotonicity and unimodality of the coefficients of Kazhdan–Lusztig polynomials for any Coxeter system [21, 43, 20]. The related geometric idea of parity sheaf from [26] has also given a precise definition of the *p-canonical basis* of a Hecke algebra for a prime $p > 0$, which led Williamson and others to construct counterexamples to Lusztig's conjecture concerning the simple characters of reductive algebraic groups and James' conjecture on the characters of irreducible modular representations for the symmetric group [51]. Soergel bimodules have also been used to categorify the Jones polynomial and the HOMFLYPT polynomial [33].

2.2 Heisenberg and Kac-Moody categorification

In parallel with the calculus of Soergel bimodules, there has been renewed interest in another family of monoidal categories and 2-categories of diagrammatic nature, namely, the Kac-Moody 2-category [45, 35] and various closely related Heisenberg categories [34, 39, 46, 10]. In particular, there has been some significant progress recently in understanding the relationship between these objects, which have also turned out to have applications to the representation theory of symmetric groups, (degenerate) affine Hecke algebras, and affine wreath product algebras, as well as to Deligne categories. For example, in [8], the Grothendieck ring of

Khovanov's original Heisenberg category has been computed, proving a conjecture of Khovanov from 2012. Heisenberg categories are important because they act naturally on many other Abelian categories, and in [9] it is expained precisely how these actions are equivalent to actions of a corresponding Kac-Moody 2-category, simplifying an old result of Rouquier known as "control by K_0", developed in [45].

As well as the Grothendieck group, all of these categories can be decategorified in other ways, including trace decategorification, and Hall algebras. The relation between the trace and the Grothendieck group of a category with suitable structure is analogous to that between K-theory and cohomology. In particular, there is a Chern character map from the trace to the Grothendieck group. Recently, mathematicians have computed the traces of categories that categorify, via the Grothendieck group decategorification, quantum groups and Heisenberg algebras [5]. This work has resulted in surprising connections to current algebras, W-algebras, and elliptic Hall algebras [15, 14]. Hall algebras themselves predate the term categorification, and they play an important role in the theory of canonical bases in quantum groups and the study of quiver representations [44]. The tremendous success of the framework of categorification has renewed interest in these classical objects, tying the former results to newer ones written in the language of categorification.

2.3 Monoidal categorification via cluster algebras

There are also natural connections between categorification and the theory of cluster algebras. Indeed, cluster algebras were originally intended as a context for computing (dual) canonical bases. In particular, the fact that the structure constants describing multiplication in these bases are positive integers already suggests that the basis elements are classes of objects in the Grothendieck ring of an appropriate monoidal category, i.e., it is the shadow of some categorification. Work of Hernandez and Leclerc [25] exploited monoidal categorifications defined in terms of representations of quantum affine algebras to understand the cluster structure on unipotent coordinate rings introduced by Geiss, Leclerc and Schrer [23]. This has been further developed by Kang, Kashiwara, Kim and Oh [27], who introduced a remarkable link between representations of quantum affine algebras, quiver Hecke algebras and cluster algebras.

3 Some open problems

There are many open problems in the areas covered by the workshop. We list here several specific problems proposed by the participants.

3.1 Monoidal categorification of cluster algebras

A monoidal category $\mathcal{C}$ is a **monoidal categorification** of the cluster algebra A if $K(\mathcal{C})$ is isomorphic to A and any cluster monomial corresponds to a real simple object in $\mathcal{C}$. In several examples of monoidal categorification of cluster algebras, mainly using the categories of finite dimensional modules over symmetric quiver Hecke algebras or quantum affine algebras, it is still an open question whether any real simple object is a cluster monomial or not. Note that this property is one of the requirements in the original formulation of the monoidal categorification of cluster algebras by Hernandez–Leclerc. This conjecture is well described in [30].

Another important open question is to give a uniform description of a categorification for any cluster algebra. At the moment, existence is only proven in some special cases, such as finite type, and the cluster algebras arising from semi-simple Lie algebras.

3.2 Rasmussen's invariant and surfaces in some four-manifolds

It is an open problem to show that the Rasmussen invariant of the torus link $T(p+q,p+q)$, with p strands oriented one way and q strands the other way ($p \geq q \geq 0$), equals $(p-q)^2 - 2p + 1$. The Rasmussen invariant is a numerical knot invariant extracted from Lees deformation of $\mathfrak{sl}(2)$ Khovanov homology. For $T(p+q,p+q)$ as above, the invariant is known when $q = 0$ (by Rasmussen) and $p = q$ (by Marengon-Manolescu-Sarkar-Willis). The latter calculation is done using Hochschild homology, and implies a genus bound for null-homologous surfaces with boundary embedded in the complex projective space. A proof of the conjecture would imply a similar bound for surfaces in any homology class in the complex projective space.

Another important question is whether or not the Rasmussen invariant be used to disprove the smooth 4-dimensional Poincaré conjecture, by finding a knot that is slice in a homotopy 4-ball, but not in the standard 4-ball. There are several classes of potential exotic 4-spheres in the literature (manifolds that are homeomorphic to the 4-sphere, but unknown to be diffeormorphic to it). One such class is given by Gluck twists, and Marengon-Manolescu-Sarkar-Willis showed that the strategy cannot work in that case (because the Rasmussen invariant has the same behavior with respect to surfaces in Gluck twists as in the standard 4-ball). However, there are other classes, e.g. those coming from nontrivial presentations of the trivial group. The challenge is to find interesting knots that are slice in such 4-balls, and to develop new methods for computing the Rasmussen invariant, especially in infinite families.

3.3 2-representation theory

Many open problems in 2-representation theory concern the classification of simple transitive 2-representations for interesting classes of 2-categories. One such interesting class is that of the Hecke 2-category, also known as Soergel bimodules. In particular, little is known about Hecke 2-categories corresponding to infinite Coxeter systems, or those in positive characteristic. In either case, current methods from finitary 2-representation theory will need to be extended significantly.

3.4 $\imath$-quantum groups

A theory of Hall algebras has been developed for quasi-split $\imath$-quantum groups (here “quasi-split corresponds to Satake diagrams with no black nodes). It is an open problem to extend the construction to $\imath$-quantum groups with the condition “quasi-split removed. Relatedly, it would be very interesting to find a general construction of a categorification of a general coideal subalgebra.

3.5 Monoidal categories

Let k be an algebraically closed field of characteristic 2, and let $\mathcal{T}$ be the category of finite-dimensional tilting representations of the algebraic group SL_2 over k. Let $\mathcal{I}_n$ be the tensor ideal of $\mathcal{T}$ generated by the n-th Steinberg representation of SL_2. Let $\mathcal{C}_n$ be the abelian monoidal envelope of the Karoubi category $\mathcal{T}/\mathcal{I}_{n+1}$, as constructed by Benson and Etingof. The categories $\mathcal{C}_n$ are symmetric k-linear tensor categories. As part of their construction, Benson and Etingof showed that there exist symmetric monoidal embeddings $\mathcal{C}_n \to \mathcal{C}_{n+1}$. Let $\mathcal{C}_\infty$ be the limit of the categories $\mathcal{C}_n$ with respect to these embeddings. An important conjecture made by Benson and Etingof states the following: consider a symmetric k-linear locally finite tensor category $\mathcal{C}$ of moderate growth (that is, for any object $X \in \mathcal{C}$ there exists an natural number M such that

$$\forall n \in \mathbb{N}, \ell(X^{\otimes n}) \leq M^n$$

where $\ell(\ )$ stands for length in $\mathcal{C}$). Then there exists an exact symmetric monoidal functor (“fiber functor”) $\mathcal{C} \to \mathcal{C}_\infty$. Other open problems in this field include constructing similar categories $\mathcal{C}_n$ for arbitrary characteristic of the base field k, and for other algebraic groups G.

3.6 Equivariantly irreducible modular representations of semisimple Lie algebras

1. Find a constructible realization of the category of suitable equivariant modules over the central reduction of the universal enveloping algebra in positive characteristic. Such a realization is known for distinguished nilpotent p-characters and for the zero p-character but not in general.

2. Extend the classification of (equivariantly) irreducible representation and the computation of their character to the De Concini–Kac form of the quantum group at roots of unity.

3.7 Categorification of Verma modules, tensor products, and the Temperley–Lieb algebra

Recent work of Vaz and others gives a categorification of the 2-parameter cyclotomic Temperley-Lieb algebra of level two, $\mathrm{End}_{sl(2)}\big(M(\lambda)\otimes V\otimes\cdots V\big)$. Some open problems include the following:

1. Give a categorification of this algebra using homotopy categories instead of derived categories, and relate it with (a quotient of) the appropriate Soergel categories.
2. Use this to give a version of annular Khovanov homology and relate it with existing constructions.
3. Generalize the above results in the following ways:
   (a) Consider tensor products of several Verma modules and several finite-dimensional irreducibles.
   (b) Extend it to other quantum Kac–Moody algebras, for example quantum $\mathfrak{gl}(n)$.
   (c) Consider tensor products of a Verma module and other irreducibles representations of $\mathfrak{sl}(2)$.

3.8 Coulomb branches

Certain geometric spaces have recently been introduced by Braverman–Finkelberg–Nakajima in order to give a mathematical definition of Coulomb branches. The category of coherent sheaves on these spaces contains a natural heart. This construction generalizes the heart consisting of perverse coherent sheaves inside the category of coherent sheaves on affine Grassmannians. Given all the mathematical and physical evidence (coming from the study of Coulomb branches of $4D$ $N=2$ gauge theories) it is natural to conjecture that this heart carries the structure of a monoidal cluster category.

4 Presentation highlights

All of the talks were highlights, we are not going to single any out above any other!

Stephen Griffeth: Harish-Chandra series for rational Cherednik algebras

I will discuss recent progress towards the Harish-Chandra classification for simple modules in category $\mathcal{O}$ of rational Cherednik algebras, and in particular discuss the extent to which Hecke algebras know the answer to the problem. Based on joint work with D. Juteau.

Nicolle Gonzalez: A categorical Boson-Fermion correspondence

Bernstein operators are vertex operators that create and annihilate Schur polynomials. These operators play a significant role in the mathematical formulation of the Boson-Fermion correspondence due to Kac and Frenkel. This correspondence bridges the actions of the infinite Heisenberg and Clifford algebras on Fock space. Cautis and Sussan conjectured a categorification of this correspondence within the framework of Khovanov's Heisenberg category. I will discuss how to categorify the Bernstein operators and settle the Cautis-Sussan conjecture.

Pedro Vaz: Categorification of Verma Modules, tensor products and the Temperley-Lieb algebra

In this talk I will review the program of categorification of Verma modules of symmetrizable quantum Kac-Moody algebras and extend it to a categorification of tensor products of a Verma module and several integrable irreducibles. In the last part of the talk I will consider the case of $\mathfrak{sl}(2)$ and explain how its categorifications are endowed with an action of the Temperley-Lieb algebra of type B with two parameters. The material presented is based upon several collaborations with Grgoire Naisse, Ruslan Maksimau and Abel Lacabanne.

Laura Rider: Exotic t-structure for coherent sheaves on a partial resolution of the nilpotent cone

In this talk, I will explain how to define an exotic t-structure for coherent sheaves on a partial resolution of the nilpotent cone. Along the way, I'll mention the special cases of exotic sheaves on the Springer resolution and perverse coherent sheaves on the nilpotent cone, along with some of their applications to geometric representation theory. I will also discuss some results on the structure of the heart. This is joint work with Paul Sobaje and Kei Yuen Chan.

Aaron Lauda: Bordered Heegaard-Floer homology, category O, and higher representation theory

The Alexander polynomial for knots and links can be interpreted as a quantum knot invariant associated with the quantum group of the Lie superalgebra $\mathfrak{gl}(1|1)$. This polynomial has been famously categorified to a link homology theory, knot Floer homology, defined within the theory of Heegaard-Floer homology. Andy Manion showed that the Ozsvath-Szabo algebras used to efficiently compute knot Floer homology categorify certain tensor products of $\mathfrak{gl}(1|1)$ representations. For representation theorists, the work of Sartori provides a different categorification of these same tensors products using subquotients of BGG category $\mathcal{O}$. In this talk we will explain joint work with Andy Manion establishing a direct relationship between these two constructions. Given the radically different nature of these two constructions, transporting ideas between them provides a new perspective and allows for new results that would not have been apparent otherwise.

Ivan Loseu: Equivariantly irreducible modular representations of semisimple Lie algebras

Let G be a semisimple algebraic group over an algebraically closed field F of very large positive characteristic. We give a combinatorial classification and find Kazhdan-Lusztig type character formulas for modules over the Lie algebra $\mathfrak{g}$ that are equivariantly irreducible with respect to an action of a certain subgroup of G whose connected component is a torus. This is a joint work with Roman Bezrukavnikov.

Jose Simental Rodriguez: Finite-dimensional representations of quantized Gieseker varieties

Quantized Gieseker varieties are algebras that quantize functions on a Gieseker moduli space. Examples include algebras of differential operators on projective space and type A spherical rational Cherednik algebras. When a quantized Gieseker variety admits a finite-dimensional representation, it admits a unique simple one and it does not admit self-extensions. In my talk, I will define the quantized Gieseker variety as a quantum Hamiltonian reduction and explain how to explicitly construct its irreducible finite-dimensional representation from the irreducible finite-dimensional representations of the type A full rational Cherednik algebra. This is joint work with Pavel Etingof, Vasily Krylov and Ivan Losev.

Ciprian Manolescu: Rasmussen's invariant and surfaces in some four-manifolds

Back in 2004, Rasmussen extracted a numerical invariant from Khovanov-Lee homology, and used it to give a new proof of Milnor's conjecture about the slice genus of torus knots. In this talk, I will describe a generalization of Rasmussen's invariant to null-homologous links in connected sums of $S^1 \times S^2$. For certain links in $S^1 \times S^2$, we compute the invariant by reinterpreting it in terms of Hochschild homology. As applications, we prove inequalities relating the Rasmussen-type invariant to the genus of null-homologous surfaces with boundary in the following four-manifolds: $B^2 \times S^2$, $S^1 \times B^3$, $\mathbb{C}P^2$, and various connected sums and boundary sums of these. We deduce that Rasmussen's invariant also gives genus bounds for surfaces inside homotopy 4-balls obtained from B^4 by Gluck twists. Therefore, it cannot be used to prove that such homotopy 4-balls are non-standard. This is based on joint work with Marco Marengon, Sucharit Sarkar, and Mike Willis.

Daniel Tubbenhauer and Vanessa Miemietz: 2-representations of Soergel bimodules

In this series of two talks we will explain the current state of the art concerning the problem of classifying "simple" 2-representations of Soergel bimodules. The first talk will introduce the asymtotic Hecke algebra and the asymptotic bicategory as well as their relationship. In particular, the asymptotic bicategories are well-understood in many cases, and this will be explained. The second part will explain the 2-representation theoretic ingredients to reduce a classification of simple transitive 2-representations of Soergel bimodules in many (most) cases to the known classification of simple transitive 2-representations of the corresponding asymptotic bicategory.

Joel Kamnitzer: Categorification via truncated shifted Yangians

The geometric Satake correspondence provides a beautiful geometric description of the representations of reductive groups, using the affine Grassmannian. In order to categorify this description, we use truncated shifted Yangians which quantize slices in the affine Grassmannian. Last year, we proved that category $\mathcal{O}$ for a truncated shifted Yangian is equivalent to a category of modules for a Khovanov-Lauda-Rouquier-Webster algebra. In this way, we obtain a categorical action on category $\mathcal{O}$'s for truncated shifted Yangians.

Myungho Kim: Monoidal categorification of cluster algebras

The notion of monoidal categorification of cluster algebras is introduced by Hernandez and Leclerc, and they provided some examples arising from the representation theory of quantum affine algebras. On the other hand, the cluster algebra structure on the unipotent quantum coordinate ring has a monoidal categorification via the representations of symmetric quiver Hecke algebras. Since there is a close connection between the representation theory of quantum affine algebras and that of symmetric quiver Hecke algebras, it is natural to ask whether one can use the results on quiver Hecke algebras to produce monoidal categorifications of cluster algebras in the category of representations over quantum affine algebras. In this talk, I will explain some results along this line of consideration. It is a joint work with S.-j. Kang, M. Kashiwara, E. Park, and S.-j. Oh.

Anthony Licata: Categorification and geometric group theory

One of the upshots of categorification constructions in representation theory is a nice stock of examples of actions of groups (e.g. braid groups) on triangulated categories. The goal of this talk will be to explain how, following an analogy with the study of mapping class groups of surfaces via Teichmuller theory, such categorical constructions can be used to study the groups themselves.

Jonathan Brundan: Heisenberg and Kac-Moody categorification

I will discuss the connections between quantum Heisenberg categories and categorified quantum groups. This is based on joint work with Alistair Savage and Ben Webster.

Sabin Cautis: Categorical structure of Coulomb branches of 4D $N = 2$ gauge theories

Coulomb branches have recently been given a good mathematical footing thanks to work of Braverman-Finkelberg-Nakajima. We will discuss their categorical structure. For concreteness we focus on the massless case which leads us to the category of coherent sheaves on the affine Grassmannian (the so called coherent Satake category). This category is conjecturally governed by a cluster algebra structure. We will describe a solution to this conjecture in the case of general linear groups and discuss extensions of this result to more general Coulomb branches of 4D $N = 2$ gauge theories. This is joint work with Harold Williams.

Zsusanna Dancso: Flow lattices and Koszul algebras

We define "q-cut" and "q-flow" lattices associated to a finite graph, arising from a categorification construction for the lattices of integer cuts and flows (which we'll introduce in the talk). We'll describe combinatorial properties of these invariants, including categorified/q-versions of some classical theorems in graph theory. Based on joint work with Tony Licata, available at `arxiv:1905.03067`.

Weiqiang Wang: Categorification of $\imath$-quantum groups via Hall algebras and quiver varieties

A quantum symmetric pair a la G. Letzter consists of a quantum group and its coideal subalgebra (called an -quantum group). A quantum group can be viewed as an example of -quantum groups associated to symmetric pairs of diagonal type, and various fundamental constructions for quantum groups (such as R-matrices and canonical bases) have been generalized to -quantum groups. In this talk, we present a realization of (universal) -quantum groups via modified Ringel-Hall algebras of $\imath$-quivers; this approach leads to PBW bases and braid group actions for $\imath$-quantum groups. Time permitting, we will also discuss a geometric realization of universal $\imath$-quantum groups via Nakajima-Keller-Scherotzke quiver varieties, which produces dual canonical bases with positivity. This is joint work with Ming Lu (Sichuan, China).

Inna Entova-Aizenbud: Abelian envelopes of rigid symmetric monoidal categories

I will define what is an abelian envelope of a fixed rigid symmetric monoidal category, in the sense of Deligne. I will also give several examples and applications, both in characteristic zero and in positive characteristic.

Amit Hazi: Ringel duality for Soergel bimodules

The category of Soergel bimodules is a well-behaved categorification of the Hecke algebra of a Coxeter group. In many characteristic 0 realizations, the indecomposable objects in this category correspond to the Kazhdan-Lusztig basis, thereby giving an explanation for the positivity of Kazhdan-Lusztig polynomials. In characteristic $p > 0$ the indecomposable objects give rise to another set of non-negative Laurent polynomials called p-Kazhdan-Lusztig polynomials, which can be used as a replacement for Kazhdan-Lusztig polynomials in modular representation theory. In this talk I will propose a non-negative replacement for inverse Kazhdan-Lusztig polynomials in positive characteristic.

Oded Yacobi: Perversity of categorical braid group actions

Let $\mathfrak{g}$ be a semisimple Lie algebra with simple roots I, and let $\mathcal{C}$ be a category endowed with a categorical $\mathfrak{g}$-action. Recall that Chuang-Rouquier construct, for every $i \in I$, the Rickard complex acting as an autoequivalence of the derived category $D^b(\mathcal{C})$, and Cautis-Kamnitzer show these define an action of the braid group $B_{\mathfrak{g}}$. As part of an ongoing project with Halacheva, Licata, and Losev we show that the positive lift to $B_{\mathfrak{g}}$ of the longest Weyl group element acts as a perverse auto-equivalence of $D^b(\mathcal{C})$. (This generalises a theorem of Chuang-Rouquier who proved it for $\mathfrak{g} = \mathfrak{sl}(2)$.) This implies, for instance, that for a minimal categorification this functor is t-exact (up to shift). Perversity also allows us to "crystallise" the braid group action, to obtain a cactus group action on the set of irreducible objects in $\mathcal{C}$. This agrees with the cactus group action arising from the $\mathfrak{g}$-crystal (due to Halacheva-Kamnitzer-Rybnikov-Weekes).

5 Outcome of the meeting

The workshop brought together established and junior researchers working in the various facets of categorification, focussing on geometric, diagrammatic, algebraic, and physical perspectives. While there are often conferences in each of these individual areas, the impact of the workshop came from its focus on the idea of categorification as a unifying concept to bring together mathematicians who do not normally meet together. Using the common language of categorification, researchers were able to discuss the important breakthroughs in their respective subfields, and explain the current outstanding open problems. The resulting cross-fertilization of ideas allowed the participants to take advantage of links between areas that have emerged in recent years. This expect that this will produce an influx of new techniques and approaches, arising from the consideration of different points of view.

Another result of the workshop was that it fostered an interaction between established mathematicians and junior ones. Since it is young as a field itself, categorification has attracted the interest of many mathematicians at the start of their career. The list of participants, which included many junior mathematicians, reflects this. These younger participants are at transitional career stages, when interacting with senior members of the mathematical community has a substantial impact, both scientifically and professionally. The participant list was selected with a view toward diversity in gender, career stage, and specialty within the domain of categorification.

The organizers would like to thank all of the participants in the workshop, and especially the speakers who provided us with a very high standard of interesting and exciting talks.

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