Singularities of Rational Inner Functions in HIGHER DIMENSIONS

by Bickel, Pascoe, Sola

Greg Knese
Washington University in St. Louis
geknese@wustl.edu

Body of talk draws from 3 papers
of Bickel, Pascoe, Sola:

1. Derivatives of rational inner functions: geometry of singularities and integrability at the boundary. PLMS 2018

2. Level curve portraits of rational inner functions. Annali della Scuola Normale Superiore di Pisa

3. Singularities of rational inner functions in higher dimensions. To appear in Amer. J. Math.

Rational Inner Functions....

- Generalize finite Blaschke products
- Are interesting in their own right because of
  - Connections to operator theory
  - Connections to stable polynomials
  - Singularities!

Rational Inner Functions (RIF)

$\phi(z) = \dfrac{q(z)}{p(z)}$ rational, analytic on

$\mathbb{D}^d = \{(z_1, \ldots, z_d) : |z_1|, \ldots, |z_d| < 1\}$

- $|\phi(z)| < 1$ for $z \in \mathbb{D}^d$
- $|\phi(z)| = 1$ for a.e. $z \in \mathbb{T}^d$

$\mathbb{T}^d = \{(z_1, \ldots, z_d) : |z_1|, \ldots, |z_d| = 1\}$

RIFs equivalent to....

<u>Real</u> <u>rational</u> <u>Pick</u> <u>functions</u>

$f : \mathbb{H}^d \to \mathbb{H}$, f rational, f real on $\mathbb{R}^d$

upper half plane

In <u>one</u> dimension,

- RIFs are Blaschke products

$$\phi(z) = c \prod_{j=1}^{N} \frac{z - a_j}{1 - \bar{a}_j z} \qquad a_j \in \mathbb{D}, \quad c \in \mathbb{T}$$

- Real rational Pick functions

$$f(z) = \frac{A(z)}{B(z)}$$

A, B real rooted w/ interlacing roots

(+ normalization)

In higher dimensions,

RIFs: $\phi = \tilde{p}/p$ (Rudin-Stout)

- $p \in \mathbb{C}[z_1, \dots, z_d]$, $p(z) \neq 0$ $z \in \mathbb{D}^d$
- $\tilde{p}(z) = z_1^{n_1} \cdots z_d^{n_d} \overline{p(1/\bar{z}_1, \dots, 1/\bar{z}_d)}$
- $\gcd(p, \tilde{p}) = 1$

$\Rightarrow$ p is a <u>stable polynomial</u>

Even better it is <u>atoral</u>.

In higher dimensions,

Real rational Pick functions

$f = \frac{A}{B} : \mathbb{H}^d \to \mathbb{H}$

$\hookrightarrow$ $A, B \in \mathbb{R}[z_1, \dots, z_d]$

$A(z), B(z) \neq 0 \quad z \in \mathbb{H}^d \cup (-\mathbb{H})^d$

$\Rightarrow$ "Real stable polynomials"

$\hookrightarrow$ • Multivariable notion of "real-rooted"

• Multivariable notion of "interlacing"

Why study RIFs?

- Any analytic $F: \mathbb{D}^d \longrightarrow \mathbb{D}$ locally uniformly approximable by RIFs. (Carathéodory $d=1$
  Radin $d>1$)
- Stable polynomials (in all their variations) are of interest in many areas of math
- One and Two variable RIFs "click" with operator-related function theory.

- See Wagner (2009) BAMS
- Borcea–Brändén
- Marcus–Spielman–Srivastava
- Analytic combinatorics

Operator-related function theory

Given RIF $\phi : \mathbb{D}^2 \to \mathbb{D}$

There exists unitary matrix $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

such that

$$\phi(z) = A + B\Delta(z)(I - D\Delta(z))C$$

$$\Delta(z) = \begin{pmatrix} z_1 I_{n_1} & \\ & z_2 I_{n_2} \end{pmatrix}.$$

- Agler
- Kummert
- Cole - Wermer
- Geronimo - Woerdeman
- Ball - Sadosky - Vinnikov

This is equivalent to a sum of squares formula.

Write $\phi = \tilde{p}/p$, then

$$\text{(SOS)} \quad |p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2)\sum_{j=1}^{n_1} |A_j(z)|^2 + (1 - |z_2|^2)\sum_{j=1}^{n_2} |B_j(z)|^2$$

RIFs in several variables can have boundary singularities....

① $\phi = \dfrac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$ $(1,1)$

$(1,1,1)$

② $\phi = \dfrac{3z_1z_2z_3 - z_1z_2 - z_2z_3 - z_3z_1}{3 - z_1 - z_2 - z_3}$

③ $\phi = \dfrac{2z_1z_2z_3 - z_1z_2 - z_3}{2 - z_1z_2 - z_3}$ $(e^{i\theta}, e^{-i\theta}, 1)$

Why study singularities of RIFs?

- Case studies for boundary regularity of bounded analytic functions.

See e.g. "Carathéodory theorem..." Agler, McCarthy, Young 2012

- Case studies for boundary regularity for more general rational functions (think analytic combinatorics)

- Boundary zeros of stable polynomials

- Uniqueness of SOS formula

(SOS) $|p(z)|^2 - |\tilde{p}(z)|^2 = (1-|z_1|^2)\sum_{j=1}^{n_1}|A_j(z)|^2 + (1-|z_2|^2)\sum_{j=1}^{n_2}|B_j(z)|^2$

governed by boundary zeros of p.

More zeros on $\mathbb{T}^2 \Rightarrow$ Fewer SOS choices.

- "Polynomials with no zeros on the bidisk" (by me)
- "Integrability and regularity of rational functions" (by me)

- Extreme points of real rational Pick functions

"Extreme points and saturated polynomials" (by me)

How to study singularities of RIFs

- Boundary regularity — non-tangential limits
  — Boundary level sets *
- Derivative integrability:
  for which $\mathfrak{p} \in [1, \infty)$ is *
  $$\frac{\partial \phi}{\partial z_j} \in L^{\mathfrak{p}}(\mathbb{T}^d)$$
- Classification of numerator ideals:
  $$\mathcal{I}_p^2 = \{ q \in \mathbb{C}[z_1, \ldots, z_d] : q/p \in L^2(\mathbb{T}^d) \}$$
  $$\mathcal{I}_p^\infty = \{ q \in \mathbb{C}[z_1, \ldots, z_d] : q/p \in L^\infty(\mathbb{T}^d) \}$$

Derivative integrability in two dimensions

Thm (BPS '18, '19) Let $\phi = \frac{\tilde{p}}{p} : \mathbb{D}^2 \to \mathbb{D}$ be an RIF.

There is a

NUMERICAL GEOMETRIC INVARIANT $K \geq 0$

associated to p such that for $1 \leq \mathfrak{p} < \infty$:

$$\frac{\partial \phi}{\partial z_1} \in L^{\mathfrak{p}}(\mathbb{T}^2) \text{ iff } \frac{\partial \phi}{\partial z_2} \in L^{\mathfrak{p}}(\mathbb{T}^2) \text{ iff } \mathfrak{p} < 1 + \frac{1}{K}$$

Contact order K

Two interpretations:

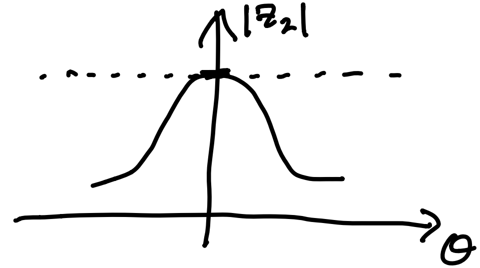
- Fastest rate branches of $Z_{\tilde{p}}$ approach $\mathbb{T}^2$: Say $\tilde{p}(1,1)=0$

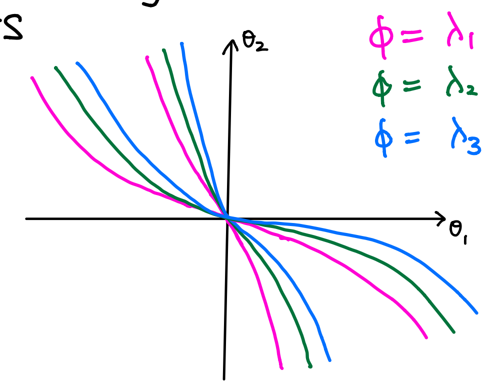
  Suppose $\tilde{p}(\zeta, w(\zeta)) = 0$

  $1-|w(\zeta)| \approx |\zeta-1|^K \qquad \zeta \in \mathbb{T}.$

- Highest order of bunching of boundary level sets

Thm (BPS '19) Two interpretations are equivalent.





$z_1 = e^{i\theta_1}$

$z_2 = e^{i\theta_2}$

Example: $\phi = \dfrac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2} = \tilde{p}/p$

① $\mathcal{Z}_{\tilde{p}} \to z_2 = \dfrac{z_1}{2z_1 - 1}$

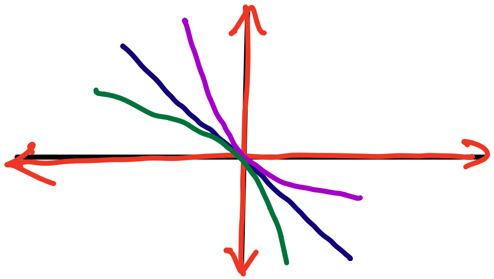
Near $z_1 = 1$, $1 - |z_2|^2 = \dfrac{|2z_1 - 1|^2 - |z_1|^2}{|2z_1 - 1|^2}$

$\approx |1 - z_1|^2 \Rightarrow K = 2$

$\Rightarrow \dfrac{\partial \phi}{\partial z_2} \in L^p(\mathbb{T}^2)$ iff $p < 3/2$.

② Boundary level sets, $\phi = 1$, $\phi = -1$

$\phi = \lambda$



Bunch together with order 2.

Higher Dimensions

Thm (BPS '20) Let $\phi = \tilde{p}/p : \mathbb{D}^d \to \mathbb{D}$ be an RIF.

Let $\delta(\phi, \zeta) = \operatorname{dist}(Z_{\tilde{p}} \cap (\{\zeta\} \times \mathbb{D}), \mathbb{T}^d)$ for $\zeta \in \mathbb{T}^{d-1}$

Set $\Omega_x = \{ \zeta \in \mathbb{T}^{d-1} : \delta(\phi, \zeta) < \frac{1}{x} \}$

Then, for $1 < p < \infty$

$$\frac{\partial \phi}{\partial z_d} \in L^p(\mathbb{T}^d) \quad \text{iff} \quad \int_1^\infty |\Omega_x| \, x^{p-2} \, dx < \infty$$

Thm (BPS '18, '19) Let $\phi = \frac{\tilde{p}}{p} : \mathbb{D}^2 \to \mathbb{D}$ be an RIF. There is a numerical geometric invariant $K^{>0}$ associated to $Z_{\tilde{p}}$ (called contact order) such that for $1 \le p < \infty$:

$$\frac{\partial \phi}{\partial z_1} \in L^p(\mathbb{T}^2) \quad \text{iff} \quad \frac{\partial \phi}{\partial z_2} \in L^p(\mathbb{T}^2) \quad \text{iff} \quad p < 1 + \frac{1}{K}$$

Let's see how this theorem $\Longrightarrow$ $d=2$ theorem

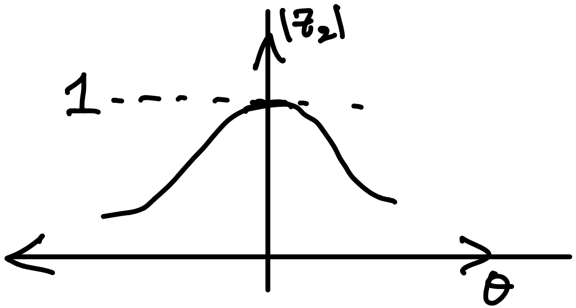
Contact order $K \implies$

$\exists$ point (WLOG) $(1,1)$ where $p(1,1)=0$

and

a branch of $Z_{\tilde{p}}$, say $z_2 = \psi(z_1)$

where $1-|\psi(z_1)| \approx |1-z_1|^K \approx \theta^K$



$z_1 = e^{i\theta}$

$\Longrightarrow$

$$\boxed{\begin{aligned} &\delta(\phi, \zeta) \approx |1-\zeta|^K \\ &\qquad \text{near } 1 \\ &\Omega_x = \{\zeta : \delta < 1/x\} \\ &\quad \supseteq \{\zeta : |1-\zeta|^K \lesssim \tfrac{1}{x}\} \\ &|\Omega_x| \approx x^{-1/K} \end{aligned}}$$

$\frac{\partial \phi}{\partial z_d} \in L^{\wp}(\mathbb{T}^d)$ iff

$$\boxed{\infty > \int_1^\infty |\Omega_x| \, x^{\wp-2}\, dx \approx \int_1^\infty x^{\wp-2-1/K}\, dx \quad \text{iff} \quad -\wp+2+\tfrac{1}{K} > 1 \quad \text{iff} \quad 1+\tfrac{1}{K} > \wp}$$

Higher Dimensions

Thm (BPS'20) Let $\phi = \tilde{p}/p : \mathbb{D}^d \to \mathbb{D}$ be an RIF.

Let $\delta(\phi, \zeta) = \mathrm{dist}(Z_{\tilde{p}} \cap (\{\zeta\} \times \mathbb{D}), \mathbb{T}^d)$ for $\zeta \in \mathbb{T}^{d-1}$

Set $\Omega_x = \{ \zeta \in \mathbb{T}^{d-1} : \delta(\phi, \zeta) < \frac{1}{x} \}$

Then, for $1 < p < \infty$

$$\frac{\partial \phi}{\partial z_d} \in L^p(\mathbb{T}^d) \quad \text{iff} \quad \int_1^\infty |\Omega_x| \, x^{p-2} \, dx < \infty$$

Lemma: Let $b(z)$ be a finite Blaschke product with zeros $\alpha_1, \dots, \alpha_N \in \mathbb{D}$

Then, for $1 \le p < \infty$

$$\int_{\mathbb{T}} |b'(z)|^p |dz| \approx \min (1 - |\alpha_j|)^{1-p}$$

Proof of Theorem:

$$\int_{\mathbb{T}^d} \left| \frac{\partial \phi}{\partial z_d}(\zeta, z_d) \right|^p = \int_{\mathbb{T}^{d-1}} \left(\int \left| \frac{\partial \phi}{\partial z_d}(\zeta, z_d) \right|^p |dz_d| \right) d\sigma(\zeta)$$

$$\approx \int_{\mathbb{T}^{d-1}} \delta(\phi, \zeta)^{1-p} \, d\sigma(\zeta)$$

$$\approx \int_0^\infty |\{ \zeta : \delta^{-1} > x \}| \, x^{p-2}$$

$$\approx \int_1^\infty |\Omega_x| \, x^{p-2} \, dx$$

Derivative Integrability $\quad \phi : \mathbb{D}^d \to \mathbb{D}$

- $d=2$:
  - Determined by single numerical invariant
  - Same for all variables
  - Boundary level sets consist locally of (unions of) analytic curves — "bunching" of branches determines K.
- $d>2$:
  - Not clear how geometry relates
  - Different integrability for different vars.
  - Boundary level sets can have several dimensional components and need not break up into smooth pieces.

Example: $\phi = \tilde{p}/p$ with different derivative integrability for different variables

---

$$p(z) = \underbrace{(2 - z_1 - z_2)}_{p_1} + \frac{1}{2} z_3 (2 z_1 z_2 - z_1 - z_2)$$

$$= p_1 \left(1 + \frac{1}{2} z_3 \phi_1\right) \qquad \phi_1 = \tilde{p}_1 / p_1$$

$$\tilde{p}(z) = \frac{1}{2} p_1 + z_3 \tilde{p}_1$$

$\frac{\partial \phi}{\partial z_3}$ bounded so $\in L^p(\mathbb{T}^3)$ $\forall p$

$$\phi = \frac{\frac{1}{2} + z_3 \phi_1}{1 + \frac{1}{2} z_3 \phi_1}$$

$\Longrightarrow$ $\frac{\partial \phi}{\partial z_1}, \frac{\partial \phi}{\partial z_2}$ have same integrability as ϕ_1.

$\in L^p(\mathbb{T}^3)$ for $p < 3/2$

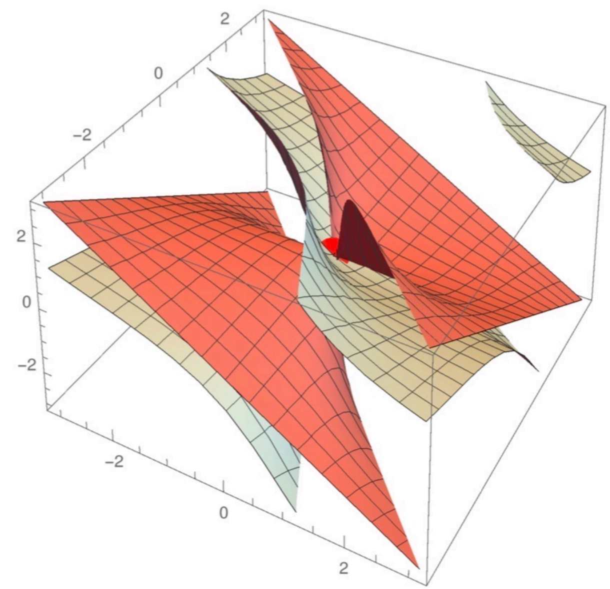
$$Z_{\tilde{p}} \cap (\mathbb{T}^2 \times \mathbb{D})$$

$$= \left\{ z_3 = -\frac{1}{2} \frac{1}{\phi_1} \right\} \cup \left\{ (1, 1, z_3) : z_3 \in \mathbb{D} \right\}$$

Example: $\phi = \dfrac{3 z_1 z_2 z_3 - z_1 z_2 - z_2 z_3 - z_3 z_1}{3 - z_1 - z_2 - z_3}$

$\dfrac{\partial \phi}{\partial z_j} \in L^p(\mathbb{T}^3)$ iff $p < 2$.

Boundary level surfaces smooth except for $\phi = -1$

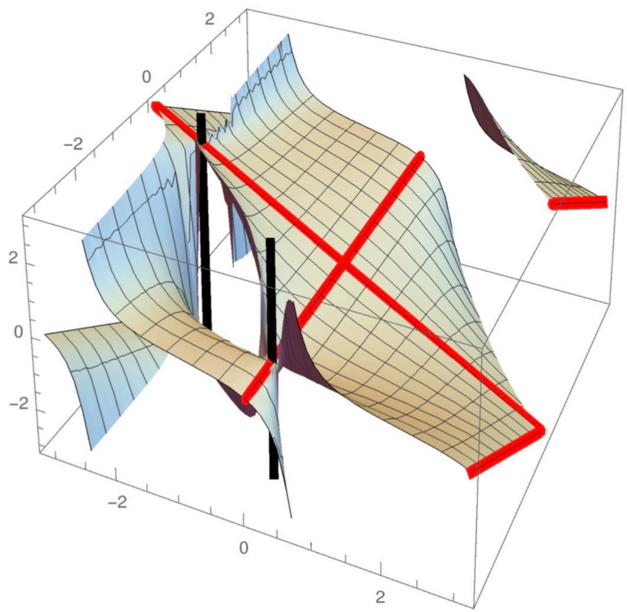


(a) Unimodular level set $\mathcal{C}_{-1}$ (salmon) with a discontinuity and a generic smooth $\mathcal{C}_\lambda$.

Many other examples

Example: $\exists$ degree $(2,1,1)$ RIF $\phi = \tilde{p}/p$ such that

- $\mathcal{Z}_p \cap \mathbb{T}^3$ consists of 3 curves
- Boundary level sets contain 2 vertical lines and a surface with a variable singularity

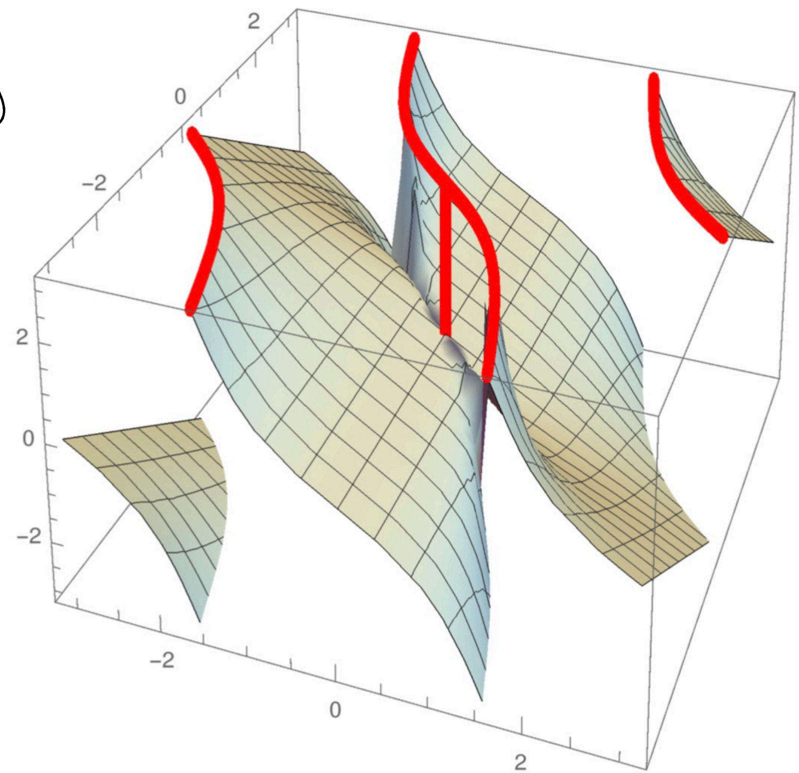


(a) $\mathcal{Z}_p \cap \mathbb{T}^3$ and a generic discontinuous $\mathcal{C}_\lambda$ ($\lambda = \exp(3i\pi/4)$) with vertical lines.

Example:

$\exists$ degree $(2,1,1)$ RIF $\phi = \tilde{p}/p$

- $\mathcal{Z}_p \cap \mathbb{T}^3$ = two curves
- Boundary level sets $(\tilde{p} - \lambda p = 0)$ all contain vertical line $\{(1,1)\} \times \mathbb{T}$ and a surface with singularity.
- $\frac{\partial \phi}{\partial z_2} \in L^p(\mathbb{T}^3)$ iff $p < 5/4$
- Integrability of $\frac{\partial \phi}{\partial z_3}$ unknown.



(a) $\mathcal{Z}_p \cap \mathbb{T}^3$ and a generic discontinuous $\mathcal{C}_\lambda$.

Main Problem:

Develop a coherent description of

- $Z_{\tilde{p}}$ near a boundary zero
- Boundary level sets of ϕ

Do singularities of RIFs interface with operator-related function theory?

¡Muchas Gracias!