Real Algebraic Geometry via Operator Theory

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Oaxaca (virtually)

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- I will also look at restrictions (via Schur complements) and extensions.
- Another important tool is the use of GNS type constructions and Hahn-Banach separation.
- Various techniques from non-commutative analysis will also be discussed.

We begin with a result that can be handled in a purely analytic fashion.

Theorem 1 ((Scalar) Fejér-Riesz Theorem).
Let $Q(\theta) = \sum_{-n}^{n} Q_k \, e^{ik\theta}$ *with coefficients in* $\mathbb{C}$ *such that* $Q(\theta) \geq 0$ *for* $\theta \in [0, 2\pi)$*. Then* $Q(\theta) = F(e^{i\theta})^* F(e^{i\theta})$ *for all* θ*, where* $F(z) = \sum_{0}^{n} F_k z^k$ *is an outer function on the unit disk with coefficients in* $\mathbb{C}$*.*

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Recall that a bounded analytic function F is *outer* if the closure of the range of F maps $H^2(\mathbb{D})$ is $H^2(\mathbb{D})$. (This is one of several equivalent characterizations.)

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For a proof, see Riesz and Sz.-Nagy. There were various generalizations to matrix valued polynomials in the 60s, culminating in an operator version due to Marvin Rosenblum.

Theorem 2 (Operator Fejér-Riesz Theorem).
Let $Q(\theta) = \sum_{-n}^{n} Q_k \, e^{ik\theta}$ *with coefficients in* $\mathcal{L}(\mathcal{H})$ *such that* $Q(\theta) \geq 0$ *for* $\theta \in [0, 2\pi)$. *Then* $Q(\theta) = F(e^{i\theta})^* F(e^{i\theta})$ *for all* θ, *where* $F(z) = \sum_{0}^{n} F_k z^k$ *is an operator-valued outer function on the unit disk with coefficients in* $\mathcal{L}(\mathcal{H})$.

Here, a bounded analytic function F is *outer* if the closure of the range of F as a multiplication operator on $H^2_{\mathcal{H}}(\mathbb{D})$ is $H^2_{\mathcal{L}}(\mathbb{D})$ for some subspace $\mathcal{L}$ of $\mathcal{H}$.

Trigonometric polynomials and Toeplitz operators

To the trigonometric polynomial

$$Q(e^{i\theta}) = \sum_{-n}^{n} A_k e^{ik\theta}$$

associate the Toeplitz operator

$$T = \begin{pmatrix} A_0 & A_{-1} & \cdots & A_{-n} & 0 & \cdots \\ A_1 & A_0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ A_n & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & & \end{pmatrix}.$$

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The polynomial Q is (strictly) positive if and only if the associated Toeplitz operator is (strictly) positive.

An elementary proof of Rosenblum's theorem

Let $Q \geq 0$, T the associated Toeplitz operator. Choose any factorization $T = F^*F$. Since T is Toeplitz, $T = S^*TS$; that is, $F^*F = (FS)^*(FS)$. So there is an isometry W such that $FS = WF$.

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$W = V \oplus U$, V a shift on $\tilde{\mathcal{L}} = \bigoplus_0^\infty \mathcal{L}$, U unitary on $\mathcal{N}$, $\overline{\operatorname{ran}}\, F = \tilde{\mathcal{L}} \oplus \mathcal{N}$ (Wold decomposition). Now an easy argument using the fact that TS^n is analytic (so commutes with S) and that $U\mathcal{N} = \mathcal{N}$ allows one to conclude that $\mathcal{N} = \{0\}$, and so $W = V$.

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Another elementary argument also shows that $\dim \mathcal{L} \leq \dim \mathcal{H}$, and that we may take F to be analytic on $\mathcal{H}$ with dense range in $H^2(\mathcal{L}) \subseteq H^2(\mathcal{H})$, and that $\deg F = n$; that is, Q has an outer factorization.

Definition 3.
Suppose

$$\tilde{A} = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$$

The *Schur complement* supported at $\mathcal{H}$ in $\tilde{A}$ (write M) is the largest positive operator X such that

$$\begin{pmatrix} A - X & B^* \\ B & C \end{pmatrix} \geq 0$$

in the sense that if Y is any other operator such that

$$\begin{pmatrix} A - Y & B^* \\ B & C \end{pmatrix} \geq 0,$$

then $M \geq Y$.

Schur complements, cont.

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This relates to the realizations that Mike Jury spoke about.

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- In particular, it can be shown that every strictly positive polynomial can be factored.
- Using a Cayley transform, it is possible to say something about rational factorization of operator valued polynomials which are positive over $\mathbb{R}^n$.
- In the scalar case, this is just a special case of either Schmüdgen's theorem or Putinar's theorem.

Schmüdgen's theorem and Putinar's theorem

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Theorem 5 (Putinar's theorem).

Let K_S be a semialgebraic set over $\mathbb{R}^d$ and suppose that M_S is archimedian. Then any polynomial which is strictly positive over K_S is in the quadratic module M_S.

Krivine's theorem

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Theorem 6 (Krivine's Striktpositivstellensatz).

For every $f \in \mathbb{R}[X]$, the following are equivalent:

1. $f(x) > 0$ *for every* $x \in K_S$, *and*
2. *there exist* $t, u \in T_S$ *such that* $(1 + t)f = 1 + u$.

Cimprič's matrix version of Krivine's theorem

Define $M_r(\mathbb{R}[X]) = M_r(\mathbb{R}) \otimes \mathbb{R}[X]$; that is $r \times r$ matrices with entries in $\mathbb{R}[X]$, equivalently, polynomials with coefficients in $M_r(\mathbb{R})$. Set $M_r(\mathbb{R}[X])^2 = \sum F_j^t F_j$, a finite sum with $F_j \in M_r(\mathbb{R}[X])$.

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Analogously, we take

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The difficult direction is (2) implies (1). This is done by induction on r and relies on the scalar version of Krivine's theorem (hence there is a non-analytic component).

Cimprič's matrix version of Krivine's theorem

The idea is to write
$$0 < F = \begin{pmatrix} f & G^T \\ G & H \end{pmatrix},$$
where $f > 0$ is scalar valued and $H \in M_{r-1}(\mathbb{R}[X])$. The Schur complement of H is $\tilde{H} = H - G(1/f)G^*$ and
$$F = K^T \begin{pmatrix} f & 0 \\ 0 & \tilde{H} \end{pmatrix} K,$$
where
$$K = \begin{pmatrix} 1 & -(1/f)G^T \\ 0 & 1_{r-1} \end{pmatrix}.$$

Clever algebraic manipulations then gets rid of the denominators.

A noncommutative Fejér-Riesz theorem

We assume G is a finitely generated discrete (hence locally compact) group G, with generators $\{1 = g_0, g_1, \ldots, g_d\}$. Define an involution by $g^* = g^{-1}$ for $g \in G$. We denote the algebraic group algebra generated by G by $\mathcal{C}$, and write S for the semigroup generated these generators.

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A "trigonometric polynomial" in $\mathcal{C} \otimes \mathcal{B}(\mathcal{H})$ is formally a finite sum over G of the form $P = \sum_g g \otimes P_g$ where $P_g \in \mathcal{B}(\mathcal{H})$ for all g. The "analytic polynomials" $\mathcal{A}$ are those trigonometric polynomials where each g is in S.

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A selfadjoint polynomial P is *positive* / *strictly positive* if for every irreducible unital $*$-representation π of G, the extension of π to the algebra $\mathcal{C} \otimes \mathcal{B}(\mathcal{H})$ by tensoring with the identity representation on $\mathcal{B}(\mathcal{H})$ (again called π), satisfies $\pi(P) \geq 0$ / $\pi(P) > 0$; where $\pi(P) > 0$ means there exists some $\epsilon > 0$ independent of π such that $\pi(P - \epsilon(1 \otimes 1)) \geq 0$.

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Letting Ω represent the set of such irreducible representations, define $\hat{P}(\pi) = \pi(P)$, and in this way think of Ω as a sort of noncommutative space on which our polynomial is defined. The Gel'fand-Raĭkov theorem ensures the existence of sufficiently many irreducible representations to separate G.

Hereditary trigonometric polynomials H are defined as those polynomials of the form $P = \sum_j w_{j1}^* w_{j2} \otimes P_j$, where $w_{j1}, w_{j2} \in S$. Write H_h for the selfadjoint elements of H.

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Any real polynomial is the sum of terms of the form $1 \otimes A$ or $w_2^* w_1 \otimes B + w_1^* w_2 \otimes B^*$, where $w_1, w_2 \in S$ and A is selfadjoint. The first of these is obviously the difference of squares. Using $w^* w = 1$ for any $w \in G$, we also have

$$w_2^* w_1 \otimes B + w_1^* w_2 \otimes B^* = (w_1 \otimes B + w_2 \otimes 1)^* (w_1 \otimes B + w_2 \otimes 1) - 1 \otimes (1 + B^* B).$$

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Sums of squares are obviously positive. In analogy with the multivariable Fejér-Riesz theorem, that strictly positive trigonometric polynomials are sums of squares.

A noncommutative Fejér-Riesz theorem

For $A, B \in \mathcal{B}(\mathcal{H})$ and $w_1, w_2 \in S$,

$$\begin{aligned} 0 &\leq (w_1 \otimes A + w_2 \otimes B)^*(w_1 \otimes A + w_2 \otimes B) \\ &\leq (w_1 \otimes A + w_2 \otimes B)^*(w_1 \otimes A + w_2 \otimes B) \\ &\qquad + (w_1 \otimes A - w_2 \otimes B)^*(w_1 \otimes A - w_2 \otimes B) \\ &= 2(1 \otimes A^*A + 1 \otimes B^*B) \\ &\leq (\|A\|^2 + \|B\|^2)(1 \otimes 1). \end{aligned}$$

Applied iteratively this shows that the order is archimedean.

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Applied iteratively this shows that the order is archimedean.

Theorem 8 (A noncommutative FR theorem).

Let G be a finitely generated discrete group, P a strictly positive trigonometric polynomial in $\mathcal{C} \otimes \mathcal{B}(\mathcal{H})$. Then P is a sum of squares of analytic polynomials.

A noncommutative Fejér-Riesz theorem

Some observations:

- $H = \mathcal{A}^*\mathcal{A}$, with positive cone C of sums of squares, and $H_h = C - C$ (ie, C is full in H).
- On $H_n := H \otimes M_n(\mathbb{C})$, set $C_n := C \otimes M_n(\mathbb{C})^+$. The same statements hold for H_n and C_n. Also $(1 \otimes 1) \otimes 1_n$ is the order unit, and for $c \in M_{n,m}(\mathbb{C})$, $c^* C_n c \subseteq C_m$. That is, we have a *matrix order* on H.
- For each n and $P \in H_n$, we define a norm by

$$\|P\|_n = \inf \left\{ t \in \mathbb{R} : \begin{pmatrix} t(1 \otimes 1) \otimes 1_n & P \\ P^* & t(1 \otimes 1) \otimes 1_n \end{pmatrix} \in C_{2n} \right\}.$$

  Complete the H_ns (write $\overline{H}_n$) with respect to these norms and take each $\overline{C}_n$ to be the closure of C_n in this norm.
- The Choi-Effros theorem then gives that $\overline{H}$ with this matrix order structure is an operator space, and so is completely isometrically order isomorphic to a concrete operator system (that is, a norm closed selfadjoint unital subspace of $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$).

A noncommutative Fejér-Riesz theorem

- H therefore generates a $*$-algebra, which can be completed to a C^*-algebra, denoted by $C^*(H)$, identified with a subalgebra of $\mathcal{B}(\mathcal{K})$. This is a subalgebra of $C^*(G)$, which we can also then identify with a subalgebra of $\mathcal{B}(\mathcal{K})$.
- For any unital representation π of G extended to $\mathcal{C} \otimes \mathcal{B}(\mathcal{H})$, and any $a \in G$, $\pi(1 \otimes a)$ is unitary. Also, the representation is automatically completely contractive.
- Suppose $H \ni P > 0$. So $P - \epsilon 1 \geq 0$. Assume $P - \epsilon 1 \notin C$. There is a linear functional $\lambda \neq 0$ on H such that $\lambda(C) \geq 0$, $\lambda(1) = 1$ and $\lambda(P) < 0$.
- λ extends to a positive linear functional on $\mathcal{B}(\mathcal{K})$ so is continuous (Kreĭn's theorem $=$ 1-d Arveson extension theorem). By the Stinespring representation theorem, it is the compression of an essential, completely contractive unital representation of $\mathcal{B}(\mathcal{K})$.
- Since π compresses to λ, $\pi(P) \ngeq 0$.

A noncommutative Fejér-Riesz theorem

- As noted, restricting to $C^*(G)$, π induces a unitary representation of G via $\pi(a_i) = \pi(a_i \otimes 1)$.
- The (irreducible) representations of G are in bijective correspondence with the essential unital $*$-representations of $C^*(G)$, so there is an irreducible unitary representation π' of G such that the corresponding representation of $C^*(G)$ has the property that $\pi'(P) \ngeq 0$, giving a contradiction.
- Taking G to be the commutative free semigroup on n generators gives the multivariable Fejér-Riesz theorem mentioned earlier, since irreducible representations of an abelian group are one dimensional.

McCullough's free NC version of the FR theorem

The set-up is very much like before, but now G is a non-commutative free group with d generators $\{g_1, \ldots, g_d\}$ and identity e, S the free semigroup with these generators.

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The hereditary words H are those of the form $v^{-1}w$ for $v, w \in S$. The set of all words of length n in S are indicated by S^n, and H^n denotes the hereditary words where $v, w \in S^n$.

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Let $U = (U_1, \ldots, U_d)$ be a d-tuple of unitary operators. For a word $w = g_{j_1} \cdots g_{j_k}$, write U_w for $U_{j_1} \cdots U_{j_k}$. If $h = v^{-1}w \in H^n$, then $U^h = (U^v)^* U^w$.

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As before, we write $\mathcal{C}$ for the algebraic group algebra. Trigonometric and analytic polynomials in $\mathcal{C} \otimes \mathcal{B}(\mathcal{H})$ are defined as above, and such a polynomial $F = \sum h_k \otimes A_k$ is said to be positive if $F(U) \geq 0$ for all d-tuples of unitaries U.

McCullough's free NC version of the FR theorem

Theorem 9 (McCullough's NC FR theorem).

Let $F \in \mathcal{C} \otimes \mathcal{B}(\mathcal{H})$ *be a positive trigonometric polynomial of degree* n *in* d *freely noncommuting variables. Set* $r = \sum_0^n d^j$*. Then there are* r *or fewer analytic polynomials* B_j *such that* $F = \sum_j B_j^* B_j$*.*

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When $d = 1$, the usual Fejér-Riesz theorem is recovered using Buerling's theorem.

McCullough's free NC version of the FR theorem

Define $\mathcal{L}(S^n)$ to be the bounded operators on the Hilbert space with orthonormal basis indexed by S^n. An operator T in this space is *Toeplitz* if $T_{v,w}$ depends only on $v^{-1}w$. Note that $T_{g_j v, g_j w} = T_{v,w}$.

The Toeplitz operators in $\mathcal{L}(S^n)$ are denoted $\mathcal{T}^n$, and those with entries in M_k are $M_k \otimes \mathcal{T}^n$. $\mathcal{T}^n$ is an operator space in $\mathcal{L}(S^n)$.

Suppose $t : H^n \to M_k$. We get a Toeplitz operator $T \in M_k \otimes \mathcal{T}^n$ with

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A version of the Caratheodory interpolation theorem allows one to extend $T \in M_k \otimes \mathcal{T}^n$ to $T' \in M_k \otimes \mathcal{T}^{n+1}$ (so ${T'}_{v,w} = T_{v,w}$ for all $v, w \in S^n$).

Continuing in this fashion, we get a positive kernel Q on a Hilbert space spanned by the elements of S tensored with $\mathbb{C}^k$.

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Define an inner product

$$\left\langle \sum x_w \otimes w, \sum y_v \otimes v \right\rangle = \sum \langle Q_{v,w} x_w, y_v \rangle,$$

then mod out null vectors and complete to a Hilbert space $\mathcal{M}$. The left regular representation of the algebraic semigroup algebra maps the generators to isometries, which can then be extended to unitaries.

Now define $V : \mathbb{C}^k \to \mathcal{K}$ by $Vx = x \otimes e$. The map

$$(1_k \otimes \varphi)(T) = (V \otimes 1_k)^* (\sum U^h \otimes A_h)(V \otimes 1_k) \geq 0$$

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To finish the proof, the Arveson extension theorem is then used to extend φ to

$$\tilde{\varphi} : M_r \to \mathcal{L}(\mathcal{H}), \quad r = \sum_0^n d^j.$$

Theorem 10 (McCullough).

Suppose that for all d-tuples S of bounded selfadjoint operators,

$$A(S) = \sum_{w \in S^{2d}} S^w \otimes A_w \geq 0.$$

Then there exists at most $r = \sum_0^n d^j$ functions $B_j = \sum_w w \otimes B_w$, such that $A(S) = B(S)^* B(S)$.

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The proof is very similar to the last, but now we work with (noncommutative) Hankel operators. The Caratheodory extension theorem is replaced by an NC flat extension theorem ($\sim$ Curto and Fialkow), and the left regular representation sends the generators to bounded selfadjoint operators.

The real version(s)

Suppose now instead that $\{x_j\}$ are the generators of S and $\{y_j\}$ are formally the adjoints of these. We consider words w which are mixtures of these letters. The algebra $\mathcal{A}$ consists of finite sums of the form $\sum p_w w$, with an obvious involution.

Let $\mathcal{A}_d$ be the subspace of all polynomials of degree at most d (so consisting of words of length at most d), and let $N(d) = \dim A_d$. Given a d-tuple $X = (X_1, \ldots, X_d)$ of operators in $\mathcal{B}(\mathbb{C}^{N(d)})$, we get a representation of the algebra by sending x_j to X_j and y_j to X_j^*. We say that p is positive if for all such d-tuples, $p(X) \geq 0$.

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Given $p \in \mathcal{A}_d$, $p \geq 0$, there are at most $N(d)$ elements $r_j \in \mathcal{A}$ such that $p = \sum_j r_j^* r_j$.

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The proof also uses Caratheodory's theorem to bound the number of terms in the sum, and a Hahn-Banach separation argument.

Helton-McCullough convex free semi-algebraic sets

The setup is as in Helton's theorem: we consider now matrix valued polynomials in freely nc variables $x_1, \ldots, x_d$ with involution defined as above, and evaluations are on d-tuples X of real symmetric $n \times n$ matrices for all n.

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For p a free symmetric polynomial, assuming $p(0) > 0$, $\mathcal{D}_p(n)$ denotes the d-tuples of $n \times n$ matrices X such that $p(X) > 0$, $\mathcal{D}_p$ the sequence of these. Say that $\mathcal{D}_p$ is convex if each $\mathcal{D}_p(n)$ is convex.

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This has the feel of an nc Kreĭn-Milman theorem. The main tool here is a version of a matricial HB theorem of Effros and Winkler.

A Striktpositivstellensatz for nc polynomials

Under the same conditions as on the last slide, say that $\mathcal{D}_p$ is *bounded* if there exists $C > 0$ such that for each $X \in \mathcal{D}_p$, $C - X_j^* X_j \geq 0$.

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For p free symmetric and $\mathcal{D}_p$ bounded, if a polynomial $q > 0$ on $\mathcal{D}_p$, then there are polynomials p_j, s_j, r_k, $t_{m,\ell}$ such that

$$q = \sum_1^N s_j^* p_j s_j + \sum_1^M r_k^* r_k + \sum t_{m,\ell}^* (1 - x_m^2) t_{m,\ell}.$$

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This is an analogue of Putinar's theorem from the commutative setting. As expected, a Hahn-Banach and a GNS construction is used. Convexity is not assumed and strict positivity is essential in general.

A Positivestellensatz for nc polynomials

If in addition, $\mathcal{D}_p$ is assumed to be convex, though not necessarily bounded, a stronger result is possible.

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Theorem 14 (Helton-Klep-McCullough Positivstellensatz).

For p free symmetric and $\mathcal{D}_p$ convex, if a polynomial $q \geq 0$ is concave on $\mathcal{D}_p$, then there are polynomials s_j, r_k, such that

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The polynomial q is concave if $-q$ is convex; that is,

$$q(tX + (1-t)Y) \geq tq(X) + (1-t)q(Y).$$

If q is scalar valued and $q(0) = 1$, the first Helton-McCullough theorem implies that concave q has the form $q(x) = 1 - \Lambda(x) - s^*(x)s(x)$, Λ a homogeneous linear polynomial and s a linear vector valued polynomial.

Trace polynomials

A *trace polynomial* is a polynomial in symmetric, nc variables along with traces of their products. For example,

$$f = x_2 x_1^2 x_2 - \mathrm{tr}(x_1 x_2) x_1^3.$$

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Klep and Schweighöfer found a connection between Positivstellensätze on classes of trace polynomials and the Connes embedding conjecture, which was recently proved to fail.


Thanks!
The End?