Dynamics of concentrated vorticities in 2d and 3d Euler flows

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New Trends in Nonlinear Diffusion:
A Bridge between PDEs, Analysis and Geometry
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The Euler equation for an incompressible inviscid fluid in $\mathbb{R}^2$ in Vorticity-stream formulation. is the problem of finding (ω, ψ) that solves the **vorticity-stream system**

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-1} \omega & \text{in } \mathbb{R}^2 \times (0, T) \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2 \end{cases} \qquad \text{(V)}$$

where

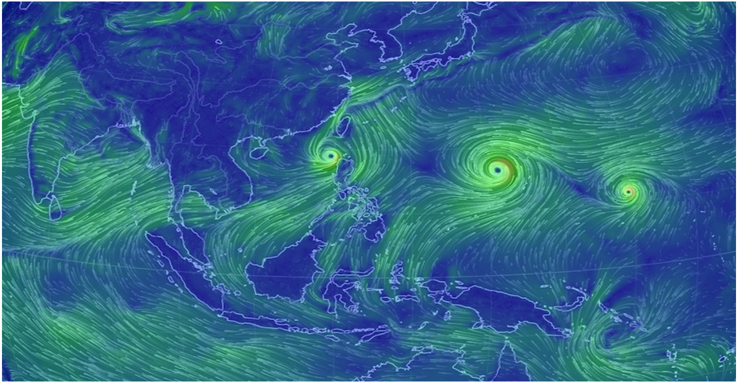
$$(-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \, f(y)\, dy.$$

$$\mathbf{u}(x,t) = \nabla^\perp \psi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y-x)^\perp}{|x-y|^2} \, \omega(y,t)\, dy$$

is the *velocity field of the fluid* (The Biot-Savart law). ω is its *vorticity*, $\omega = \nabla \times \mathbf{u} = \partial_x u_2 - \partial_y u_1$.

- Global existence and uniqueness in $L^\infty(\mathbb{R}^2)$ of the initial value problem (V) is known: Wolibner (1933), Yudovich (1963). Solutions are regular if ω_0 is.

We are interested in describing the evolution of solutions to system (V) whose vorticities $\omega(x, t)$ are **very concentrated** around a finite number of points.



$$\omega(x,t) \approx \sum_{j=1}^{k} 8\pi\kappa_j \delta(x - \xi_j(t)), \quad \Psi(x,t) \approx \sum_{j=1}^{k} 4\kappa_j \log \frac{1}{|x - \xi_j(t)|}$$

Analysis of solutions with highly concentrated vorticities:
A mathematical subject with a long history: it traces back to Helmholtz (1858), Kirchhoff (1876), Routh (1881), Lagally (1921) C.C. Lin (1941).

Formal N-vortex singular solutions (ω^s, Ψ^s) of (V):

$$\omega^s(x,t) = \sum_{j=1}^{N} 8\pi\kappa_j\,\delta(x-\xi_j(t)),$$

where $\delta(x)$ is the Dirac mass at 0, $\kappa_j \in \mathbb{R}$, $\xi_j : [0,T] \to \Omega$.
Since $\Psi^s = (-\Delta)^{-1}\omega^s$, we must have

$$\Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j \Gamma(x-\xi_j(t)), \quad \Gamma(x) = 4\log\frac{1}{|x|}.$$

Formally we compute

$$\begin{aligned}\omega_t^s + \nabla^\perp \Psi^s \cdot \nabla \omega^s \; &= \; -\sum_{j=1}^{N} 8\pi\kappa_j \nabla\delta(x-\xi_j)\cdot\dot{\xi}_j, \\ &+ \sum_{i,j=1}^{N} 8\pi\kappa_i\kappa_j \nabla^\perp\Gamma(x-\xi_i)\cdot\nabla\delta(x-\xi_j) \\ &= 8\pi\sum_{j=1}^{N}[-\kappa_j\dot{\xi}_j + \nabla_x^\perp\big(8\pi\sum_{i\neq j}\kappa_i\kappa_j\Gamma(x-\xi_i)\big)]\cdot\nabla\delta(x -\end{aligned}$$

We use $\Gamma(x)$, $\delta(x)$ are "radial": $\nabla^\perp\Gamma(x-\xi_j)\cdot\nabla\delta(x-\xi_j)=0$. Thus $(\omega^s.\Psi^s)$ is a "solution" of Problem (V) if and only if $(\xi_1,\ldots,\xi_N)$ solves the planar N-body problem

$$\boxed{\dot{\xi}_j(t) = \sum_{i\neq j} 4\kappa_i \frac{(\xi_i(t)-\xi_j(t))^\perp}{|\xi_i(t)-\xi_j(t)|^2}, \quad j=1,\ldots,N.} \qquad (K)$$

A Natural question: Do solutions whose initial vorticity is highly concentrated around a finite set of points have "vorticity packets" evolving by a dynamics approximated by (K)?
We define $\omega_{0\varepsilon}$ and $\Psi_{0\varepsilon}$ explicit ε-regularizations of the singular solution

$$\omega^s(x,t) = \sum_{j=1}^{N} 8\pi\kappa_j \delta(x-\xi(t)), \quad \Psi^s(x,t) = \sum_{j=1}^{N} 2\kappa_j \log \frac{1}{|x-\xi_j(t)|^2}$$

$$\boxed{\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{N} 2\kappa_j \log \frac{1}{|x-\xi_j(t)|^2+\varepsilon^2}}$$

$$\boxed{\omega_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{8}{(1+|y|^2)^2}\ .}$$

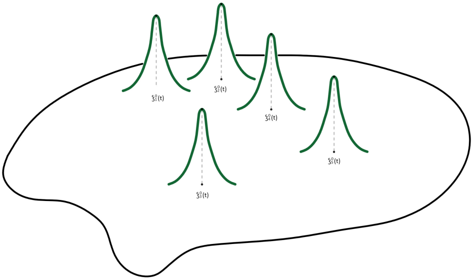
We have $(-\Delta)^{-1}\omega_{0\varepsilon} = \Psi_{0\varepsilon}$ and $\int_{\mathbb{R}^2} U_0 = 8\pi$. We get

$$\omega_{0\varepsilon} \rightharpoonup \sum_{j=1}^{N} \kappa_j \delta(x - \xi_i(t)), \quad \frac{1}{|\log \varepsilon|} |\nabla^{\perp}\Psi_{0\varepsilon}|^2 \rightharpoonup \sum_{j=1}^{N} 32\pi\kappa_j^2 \delta(x - \xi_j(t)).$$

We prove that for a given colisionless solution of the system

$$\boxed{\dot{\xi}_j(t) = \sum_{i \neq j} \kappa_i 4 \frac{(\xi_i(t) - \xi_j(t))^{\perp}}{|\xi_i(t) - \xi_j(t)|^2}, \quad j = 1, \ldots, N.} \qquad (K)$$

there is a solution of system (V) that differs little from $(\Psi_{0\varepsilon}, \omega_{0\varepsilon})$.



Theorem (Dávila, del Pino, Musso, Wei, ARMA 2020)

Let $\xi(t)$ *be a colisionless solution of* (K) *in* $[0, T]$. *There exists a solution* $(\omega_\varepsilon, \Psi_\varepsilon)$ *of Problem* (V) *of the form*

$$\omega_\varepsilon(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right) + \phi(x,t)$$

$$\Psi_\varepsilon(x,t) = \sum_{j=1}^{k} \kappa_j \log \frac{1}{(\varepsilon^2 + |x-\xi_j|^2)^2} + \psi(x,t)$$

where for some $0 < \sigma < 1$ *and all* $(x,t) \in \mathbb{R}^2 \times (0,T)$ *we have*

$$|\phi(x,t)| \ \le \ \varepsilon^{\sigma} \sum_{j=1}^{k} \frac{1}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right),$$

$$|\psi(x,t)| + \varepsilon |D_x \psi(x,t)| \ \le \ \varepsilon^2.$$

In particular:

$$\omega_\varepsilon \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta(x-\xi_i) \quad \frac{1}{|\log\varepsilon|}|\nabla\Psi_\varepsilon|^2 \rightharpoonup \sum_{j=1}^{k} \kappa_j^2 \delta(x-\xi_j).$$

A prior result along these lines, Marchioro and Pulvirenti (1993).

Ingredients in the construction:

• Improvement of the approximation in powers of ε using elliptic and transport equations.

• Setting up the problem as a coupled system of inner problems near the singularities and and an outer problem more regular (the inner-outer gluing scheme)

• A priori estimates to solve by a continuation (degree) argument.

Improving the approximation. Let $\Gamma_0(y) := \log \frac{8}{(1+|y|^2)^2}$,

$$\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \kappa_j \Gamma_0\left(\frac{x-\xi_j(t)}{\varepsilon}\right) - \frac{\kappa_j}{8\pi}\log 8\varepsilon^2$$

We want to solve the equation $E(\omega,\Psi) = 0$, where

$$E(\omega,\Psi) := \omega_t + \nabla_x^{\perp}\Psi\cdot\nabla_x\omega, \quad -\Delta_x\Psi = \omega.$$

Near $\xi_j(t)$ write $y = \frac{x-\xi_j(t)}{\varepsilon}$. We look for a solution of the form

$$\Psi = \Psi_{0\varepsilon}(x,t) + \kappa_j\psi(y,t), \quad \omega = \frac{\kappa_j}{\varepsilon^2}U_0(y) + \frac{\kappa_j}{\varepsilon^2}\phi(y,t).$$

In terms of the y-variable we get the expression

$$\boxed{\begin{aligned} \varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t \; + \; (-\varepsilon\dot{\xi} + \nabla_y^\perp \Psi_{0\varepsilon} + \kappa_j \nabla_y^\perp \psi)\cdot \nabla_y (U_0 + \phi), \\ -\Delta_y \psi =& \phi \end{aligned}}$$

We have

$$\Psi_{0\varepsilon}(x,t) = \kappa_j \Gamma_0(y) + \varphi(x) + O(\varepsilon^2) + constant, \quad y = \frac{x-\xi_j}{\varepsilon},$$

$$\varphi(x) = \sum_{i\neq j} \kappa_i \Gamma(x-\xi_i).$$

By assumption $\dot{\xi}_j = \nabla_x^\perp \varphi(\xi_j)$, hence we get

$$-\varepsilon\dot{\xi}_j + \nabla_y^\perp \Psi_{0\varepsilon}(\xi_j + \varepsilon y) = \kappa_j \nabla^\perp(\Gamma_0 + \mathcal{R})$$

with $\mathcal{R} = O(\varepsilon^2 |y|^2)$.

$$
\begin{aligned}
\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t \;+\; \kappa_j \nabla_y^{\perp}(\Gamma_0(y)+\mathcal{R}+\psi)\cdot\nabla_y(U_0+\phi),,\\
-\Delta_y\psi =& \phi\\
\mathcal{R} =& O(\varepsilon^2|y|^2)
\end{aligned}
$$

Let $f(u) = e^u$. Since $U_0 = f(\Gamma_0)$ we find

$$
\begin{aligned}
\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t \;-\; \kappa_j \nabla_y^{\perp}\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi)\\
&+\kappa_j\nabla^{\perp}\mathcal{R}\cdot\nabla U_0 + \kappa_j\nabla^{\perp}\mathcal{R}\nabla\phi + \nabla^{\perp}\psi\nabla\phi.
\end{aligned}
$$

The 0-error term:

$$
\varepsilon^4 E(\omega_{0\varepsilon},\Psi_{0\varepsilon}) = \nabla^{\perp}\mathcal{R}\cdot\nabla U_0 = O(\varepsilon^2|y|^{-4}).
$$

We obtain a reduction in the error by solving the elliptic equation

$$
-\nabla_y^{\perp}\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + \nabla^{\perp}\mathcal{R}\cdot\nabla U_0 = 0
$$

After sufficiently improving the approximation we solve the problem by a continuation (degree) argument, which near each ξ_j reads approximately as

$$\boxed{\begin{aligned} \varepsilon^2\phi_t - \nabla^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0 \\ -\Delta\psi = \phi \quad \text{in } \mathbb{R}^2\times[0,T] \end{aligned}}$$

with $E = O(\varepsilon^5\rho^{-3})$.

A basic ingredient: A priori estimates under initial datum zero and $\int_{B_{\delta/\varepsilon}} y\,\phi(y,t)\,dy = 0, \quad \int_{\mathbb{R}^2}\phi(y,t)\,dy = 0$

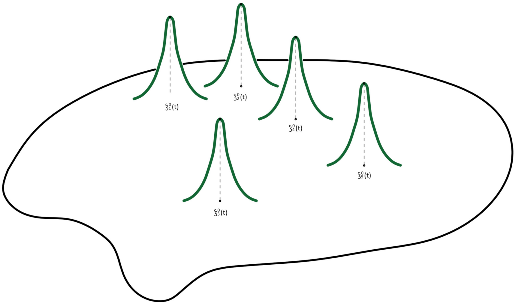
$$\|\phi(\cdot,t)U_0^{-\frac12}\|_{L^2}^2 \lesssim \varepsilon^{-2}|\log\varepsilon| \sup_{t\in[0,T]}\|EU_0^{-\frac12}\|_{L^2}^2$$

This allows a fixed point scheme to work when $E = O(\varepsilon^5\rho^{-3})$.

The **generalized surface quasigeostrophic equation (SQG)**
Let $\frac{1}{2} < s < 1$.

$$\begin{cases} \omega_t + \nabla^{\perp}\psi \cdot \nabla\omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-s}\omega & \text{in } \mathbb{R}^2 \times (0, T), \end{cases}$$

$$(-\Delta)^{-s} f(y) = c_s \int_{\mathbb{R}^2} \frac{1}{|x-y|^{2-2s}} f(y)\, dy$$



$$\boxed{\dot{\xi}_j(t) = \sum_{i\neq j} \kappa_i d_s \frac{(\xi_i(t)-\xi_j(t))^\perp}{|\xi_i(t)-\xi_j(t)|^{4-2s}}, \quad j=1,\dots,N.} \qquad (K_s)$$

Theorem (M. D., Antonio Fernandez)

For a colisionless solution $\xi(t)$ of the N-body problem (K_s) there exists a solution of (SQG) such that

$$\omega(x,t) \approx \sum_{j=1}^{N} k_j \frac{1}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{c_s}{(1+|x|^2)^{1+s}}$$

The proof is substantially harder.

Special case: A travelling wave solution for the case of a travelling vortex pair: Ao, Dávila, del Pino, Musso, Wei (TAMS, 2021).

Nearly singular solutions for Euler in $\mathbb{R}^3$?

Open question: Solutions with concentrated vorticities near curves (filaments): *the Vortex filament conjecture* (Helmholtz, Da Rios, Levi-Civita 1858-1906-1931).

We consider the Euler equation in $\mathbb{R}^3$ in stream-vorticity formulation

$$\begin{cases} \omega_t + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \quad , \\ \mathbf{u} = \nabla \times \psi, \quad \psi(x,t) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \dfrac{x-y}{|x-y|^2} \times \omega(y,t) dy. \end{cases} \tag{V}$$

($\omega = \nabla \times \mathbf{u}$ in $\mathbb{R}^3$). We want to find solutions with vorticity concentrated on a time evolving curve (filament) $\Gamma(t)$ parametrized by arclength as $\gamma(s,t)$ in $\mathbb{R}^3$.

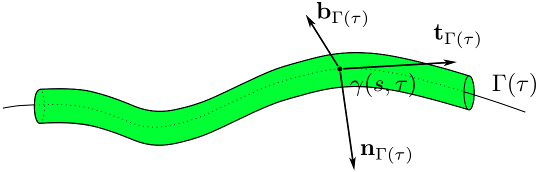
Vortex filament dynamics. (After Helmholtz and Kelvin) is a solution $\omega_\varepsilon(x,t)$ of (V) concentrated in a tube radius ε so that

$$\omega_\varepsilon(\cdot,t) \approx c\delta_{\Gamma(t)}\mathbf{t}_{\Gamma(t)} \quad \text{as } \varepsilon \to 0,$$

$\mathbf{t}_{\Gamma(t)}$ tangent vector field, $\delta_{\Gamma(t)}$ the uniform curve Dirac measure.
1904, Da Rios formal law: Letting $\tau = t|\log\varepsilon|$, $\gamma(s,\tau)$ parametrization by arclength of $\Gamma(\tau)$, κ curvature, then

$$\boxed{\gamma_\tau = \frac{c}{4\pi}(\gamma_s \times \gamma_{ss}) = \frac{c}{4\pi}\kappa\mathbf{b}_{\Gamma(\tau)}},$$

$\mathbf{b}_{\Gamma(\tau)}$ binormal vector. This is the *binormal flow of curves*.



The *vortex filament conjecture*:

Let $\Gamma(\tau)$ *be a solution curve of the binormal flow defined in* $[0, T]$ *for some* $c > 0$, $T > 0$. *For each* $\varepsilon > 0$ *there exists a smooth solution* $\omega_\varepsilon(x, t)$ *to (V) satisfying in the distributional sense,*

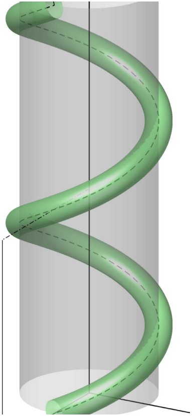
$$\omega_\varepsilon(\cdot, \frac{\tau}{|\log \varepsilon|}) \rightharpoonup c\delta_{\Gamma(\tau)}\mathbf{t}_{\Gamma(\tau)} \quad \text{as } \varepsilon \to 0, \quad \text{for all} \quad 0 \le \tau \le T.$$

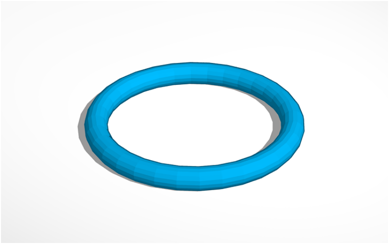
Natural: To look for a solution of the form

$$\omega_\varepsilon(x, \tau) = \frac{1}{\varepsilon^2} U_0\left(\frac{z}{\varepsilon}\right)\mathbf{t}_{\Gamma(\tau)} + o(1), \quad x = \gamma(\tau, s) + z_1\mathbf{b}_{\Gamma(\tau)} + z_2\mathbf{n}_{\Gamma(\tau)},$$

This statement is only known for special curves associated to travelling wave solutions: the thin vortex ring first found by Fraenkel, and recently a helicoidal filament.

Examples: a helix whose horizontal section rotates at a constant angular speed or a vertically translating circle are solutions of the bi-normal flow of curves.





Solutions $\vec{\omega}(x, y, z, t)$ of $3d$-Euler with Helicoidal symmetry can be obtained from a scalar function $w(x + iy, t)$ in the form

$$\vec{\omega}(x, y, z, t) = w(e^{-iz}(x + iy), t) \begin{bmatrix} i(x + iy) \\ b \end{bmatrix}$$

where

$$\begin{cases} |\log \varepsilon| w_t + \nabla^\perp \psi \cdot \nabla w = 0 \\ \qquad -\nabla \cdot (K \nabla \psi) = w \end{cases}$$

$$K(x, y) = \frac{1}{\kappa^2 + x^2 + y^2} \begin{pmatrix} \kappa^2 + y^2 & -xy \\ -xy & \kappa^2 + x^2 \end{pmatrix}$$

Rotating helicoidal solutions:

$$w(x + iy, t) = w(e^{i\alpha t}(x + iy)), \quad \psi((x + iy), t) = \psi(e^{i\alpha t}(x + iy)).$$

The problem reduces to the elliptic equation

$$\boxed{-\nabla \cdot (K \nabla \psi) = f(\psi - \frac{\alpha}{2} |\log \varepsilon| (x^2 + y^2)) = w \quad \text{in } \mathbb{R}^2}$$

Special case $f(u) = \varepsilon^2 e^u$. we prove:

Theorem (Dávila, del Pino, Musso, Wei, Arxiv 2021)

There exists a solution ψ_ε *to the equation*

$$-\nabla \cdot (K \nabla \psi) = \varepsilon^2 e^{\psi + \lambda(x^2+y^2)} \quad \textit{in } \mathbb{R}^2$$

such that $\varepsilon^2 e^{\psi - \frac{\alpha}{2}|\log \varepsilon|(x^2+y^2)} \rightharpoonup 8\pi\delta_{(x_0,0)}$, $x_0 > 0$, *for a suitable choice of* α.

α is precisely the number that makes the "rotating helix"

$$\gamma(s,t) = \begin{pmatrix} e^{i\left(\frac{s}{\sqrt{b^2+x_0^2}} - \alpha t\right)}(x_0 + iy_0) \\ \frac{bs}{\sqrt{b^2+x_0^2}} \end{pmatrix}$$

a solution of the binormal flow

Another known solution of the binormal flow that does not change its form in time is the vortex ring.

Axisymmetric Euler no-swirl: Cylindrical coordinates

$$\omega(r,z,t) = W(r,z,t)(-y,x).$$

After rescaling time, we get

$$\begin{cases} |\log\varepsilon| r W_t + \nabla^{\perp}(r^2\psi)\nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz}) := -\Delta_5\psi = W \\ \psi_r(0,z,t) = 0. \end{cases}$$

Fraenkel's exact traveling ring solutions (1970-1972): $W = W(r, z - \alpha t)$. It solves

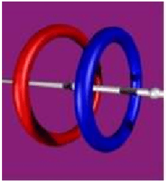
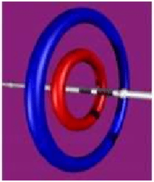
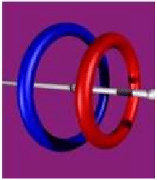
$$-\alpha r|\log\varepsilon| W_z + \nabla^{\perp}(r^2\psi)\nabla(W) = 0$$

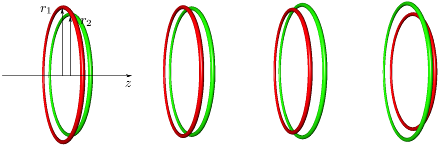
$$-\Delta_5\psi = W$$

Take $W = F(r^2(\psi - \alpha|\log\varepsilon|))$.

Leapfrogging Vortex-Rings

Helmholtz 1858: We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.



Aim: Mathematically justify the leap-frogging dynamics for the 3d axisymmetric Euler flow without swirl.

$$\begin{cases} |\log\varepsilon| r W_t + \nabla^\perp(r^2\psi)\nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz}) = W \\ \psi_r(0,z,t) = 0 \end{cases}$$

$$W(r,z,t) = \sum_{j=1}^{2} r(a_j)^{-1}\frac{1}{\varepsilon_j^2} U_0\left(\frac{x-a_j}{\varepsilon_j}\right)$$

where $a_j = a_j(t), \varepsilon_j = \varepsilon_j(t),\ j = 1,2$ and $\sqrt{r(a_j)}\varepsilon_j(t) = \varepsilon$,

Theorem [Dávila, del Pino, Musso, Wei, 2021] Let $a(t) = (a_1(t), \ldots, a_N(t))$ be a colisionless solution of the system

$$\begin{cases} \dot{b}_i(t) = \sum_{j\neq i} \frac{(b_i-b_j)^\perp}{|b_i-b_j|^2} - \frac{r(b_i)}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ b_i(0) = b_i^0 \end{cases}$$

$$a_i(t) = \left(r_0 + \frac{r(b_i(t))}{\sqrt{|\log \varepsilon|}}, z_0 + \frac{t}{r_0} + \frac{z(b_i(t))}{\sqrt{|\log \varepsilon|}} \right)$$

in $(0, T)$. Then there exists a solution W_ε of 3D axisymmetric Euler flow (without swirl) of the form

$$\boxed{W_\varepsilon(x,t) = \sum_{j=1}^{N} \frac{1}{r(a_j)\varepsilon_j^2} U_0 \left(\frac{(r,z) - a_j}{\varepsilon_j} \right) + o(1)}$$

$\varepsilon = \sqrt{r(a_j(t))}\varepsilon_j(t).$

Thanks for your attention