Parametrised moduli spaces of surfaces as infinite loop spaces

joint work with Florian Kranhold and Jens Reinhold

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A formulation of the Madsen-Weiss theorem

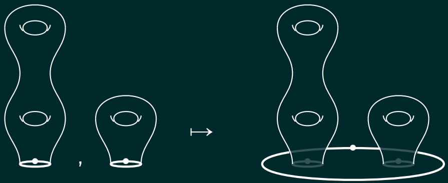
Let $\mathfrak{M}_{g,1}$ denote the moduli space of compact Riemann surfaces of type $\Sigma_{g,1}$, i.e. of genus $g \geqslant 0$ with one *parametrised* boundary component.
$\mathfrak{M}_{g,1}$ is a classifying space for the mapping class group $\Gamma_{g,1} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,1}, \partial\Sigma_{g,1}))$:

$$\mathfrak{M}_{g,1} \simeq B\Gamma_{g,1}.$$

Gluing two Riemann surfaces of type $\Sigma_{g,1}$ and $\Sigma_{g',1}$ using a disc with two holes gives a surface of type $\Sigma_{g+g',1}$. The space

$$\mathfrak{M}_{*,1} := \coprod_{g\geqslant 0} \mathfrak{M}_{g,1}$$

is (up to homotopy) a topological monoid (in fact it is an E_2-algebra).



Theorem (Madsen, Weiss)

There is an equivalence of loop spaces

$$\Omega B\mathfrak{M}_{*,1} \simeq \Omega^\infty \mathrm{MTSO}(2).$$

What about free loop spaces of moduli spaces?

$$\mathfrak{M}_{*,1} := \coprod_{g \geqslant 0} \mathfrak{M}_{g,1} \qquad \Omega B\mathfrak{M}_{*,1} \simeq \Omega^\infty \mathrm{MTSO}(2)$$

Consider now the free loop spaces $\Lambda\mathfrak{M}_{g,1} := \mathrm{map}(S^1, \mathfrak{M}_{g,1})$ for $g \geqslant 0$.
Λ-wise multiplication makes

$$\Lambda\mathfrak{M}_{*,1} := \coprod_{g \geqslant 0} \Lambda\mathfrak{M}_{g,1}$$

also into a topological monoid (in fact an E_2-algebra).

Question

What does $\Omega B\Lambda\mathfrak{M}_{*,1}$ look like?

- In fact, $\Lambda\mathfrak{M}_{*,1}$ is not only an E_2-algebra, but also an algebra over Tillmann's surface operad $\mathcal{M}$. A result by Tillmann implies that $\Omega B\Lambda\mathfrak{M}_{*,1}$ is an Ω^∞-space.
- There are maps of $\mathcal{M}$-algebras

$$\mathfrak{M}_{*,1} \xrightarrow{\mathrm{const}} \Lambda\mathfrak{M}_{*,1} \xrightarrow{\mathrm{ev}_1} \mathfrak{M}_{*,1}$$

with composition the identity. Therefore $\Omega^\infty \mathrm{MTSO}(2)$ is a factor of $\Omega B\Lambda\mathfrak{M}_{*,1}$.

Main result, and the work of BBPTY

Theorem (B., Kranhold, Reinhold)

For a suitable topological space $\mathfrak{X}$, there is an equivalence of loop spaces

$$\Omega B\Lambda\mathfrak{M}_{*,1} \simeq \Omega^\infty \mathrm{MTSO}(2) \times \Omega^\infty \Sigma^\infty_+ \mathfrak{X}.$$

So far we only know $\Omega B\Lambda\mathfrak{M}_{*,1} \simeq \Omega^\infty \mathrm{MTSO}(2) \times \Omega^\infty$ *???.*

Tillmann's result was recently improved by Basterra, Bobkova, Ponto, Tillmann and Yeakel to the setting of (monochromatic) operads $\mathcal{O}$ with homological stability (OHS). For example, Tillmann's surface operad $\mathcal{M}$ is an OHS.
If $\mathcal{O}$ is an OHS, BBPTY give a quite direct way to compute ΩBY for a $\mathcal{O}$-algebra Y: for example, if $Y = F^{\mathcal{O}}(X)$ is the *free* $\mathcal{O}$-algebra over an unpointed space X, then

$$\Omega BF^{\mathcal{O}}(X) \simeq \Omega B\mathcal{O}(0) \times \Omega^\infty \Sigma^\infty_+ X,$$

where $\mathcal{O}(0)$ is the initial $\mathcal{O}$-algebra (in the case $\mathcal{O} = \mathcal{M}$, we have $\mathcal{O}(0) \simeq \mathfrak{M}_{*,1}$).

Naive conjecture

There is a space $\mathfrak{X}$ such that $\Lambda\mathfrak{M}_{*,1} \simeq F^{\mathcal{M}}(\mathfrak{X})$.

The previous conjecture turns out to be wrong, but not completely wrong.

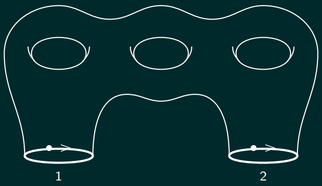
Generic moduli spaces $\mathfrak{M}_{g,n}$

In order to understand the structure of $\Lambda\mathfrak{M}_{*,1}$ as a $\mathcal{M}$-algebra, we need to consider all surfaces of type $\Sigma_{g,n}$, for all $g \geqslant 0$ and $n \geqslant 1$.

The moduli space $\mathfrak{M}_{g,n}$ contains equivalence classes of Riemann surfaces $\mathcal{S}$ of type $\Sigma_{g,n}$ with *ordered* and *parametrised* boundary components, i.e. $\mathcal{S}$ is endowed with a diffeomorphism

$$\partial\mathcal{S} \cong \{1, \dots, n\} \times S^1$$

compatible with boundary orientation.



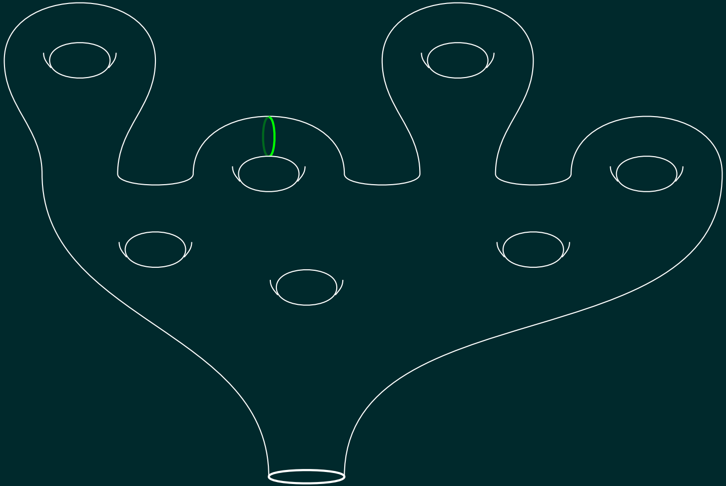
$\mathfrak{M}_{g,n}$ is a classifying space for the mapping class group $\Gamma_{g,n} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n}))$.
For $g \geqslant 0$ and $n \geqslant 1$ we homotopy equivalences

$$\Lambda\mathfrak{M}_{g,n} \simeq \Lambda B\Gamma_{g,n} \simeq \coprod_{[\varphi]\in\mathrm{Conj}(\Gamma_{g,n})} BZ(\varphi, \Gamma_{g,n})$$

- $\mathrm{Conj}(\Gamma_{g,n})$ is the set of conjugacy classes of $\Gamma_{g,n}$;
- $Z(\varphi, \Gamma_{g,n})$ is the centraliser of φ in $\Gamma_{g,n}$.

The homotopy type of $\Lambda\mathfrak{M}_{*,1}$ depends on the groups $Z(\varphi, \Gamma_{g,1})$ for $\varphi \in \Gamma_{g,1}$; to describe these centralisers we will need *all groups* $\Gamma_{g,n}$, also for $n > 1$.

Examples of mapping classes and their centralisers



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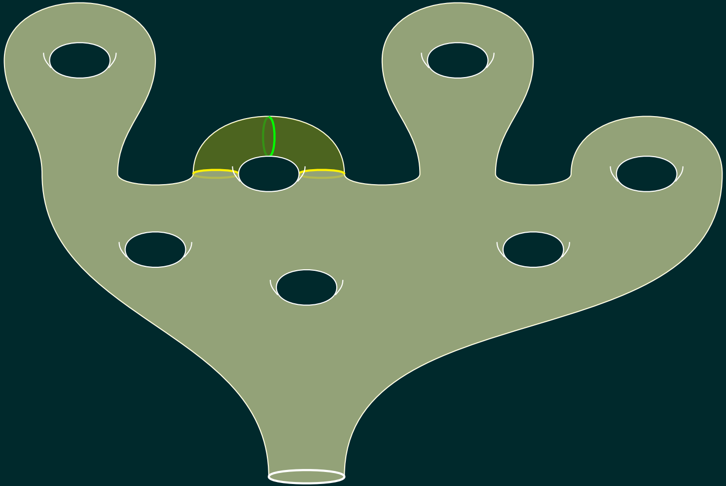

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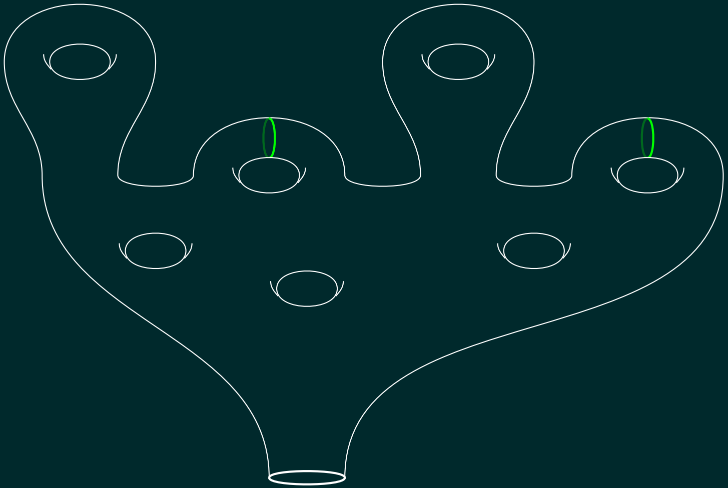

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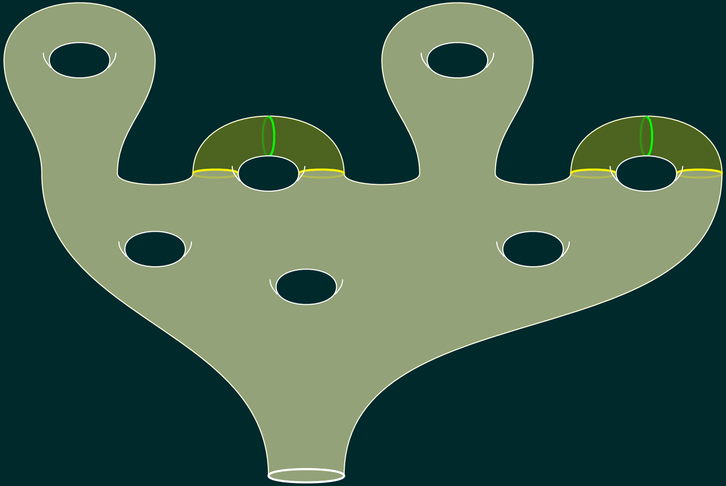

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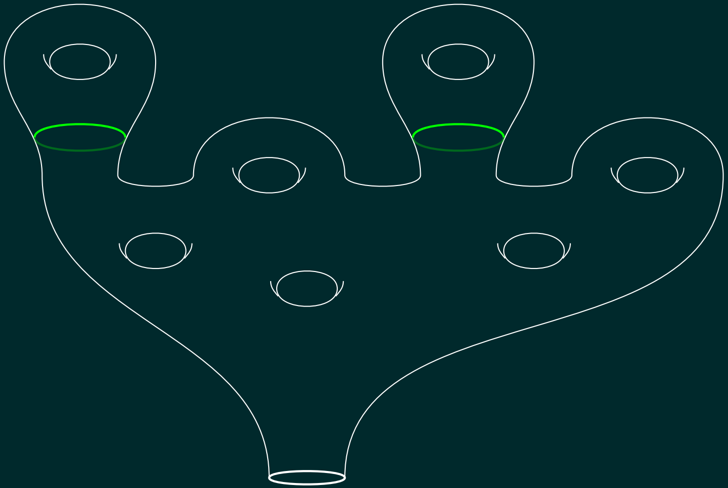

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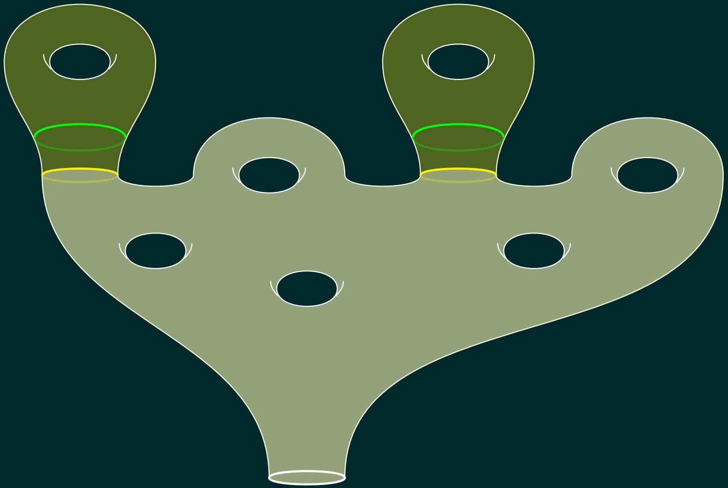

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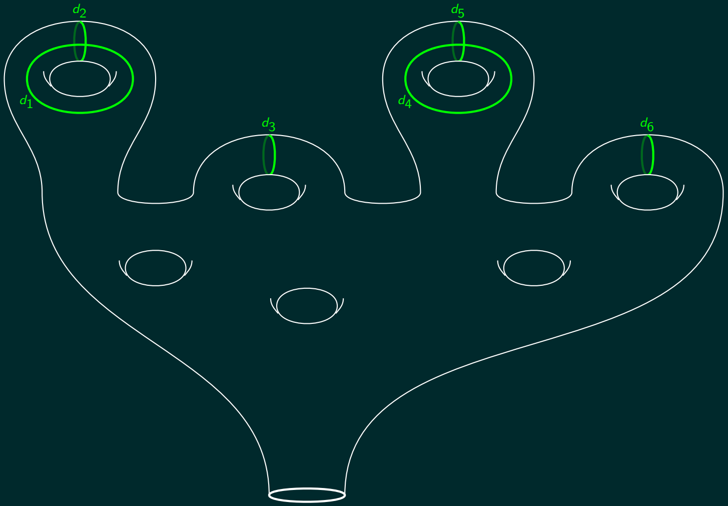

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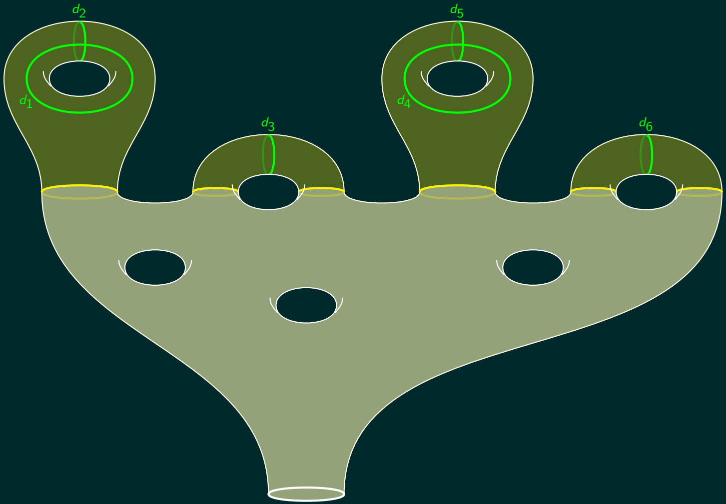

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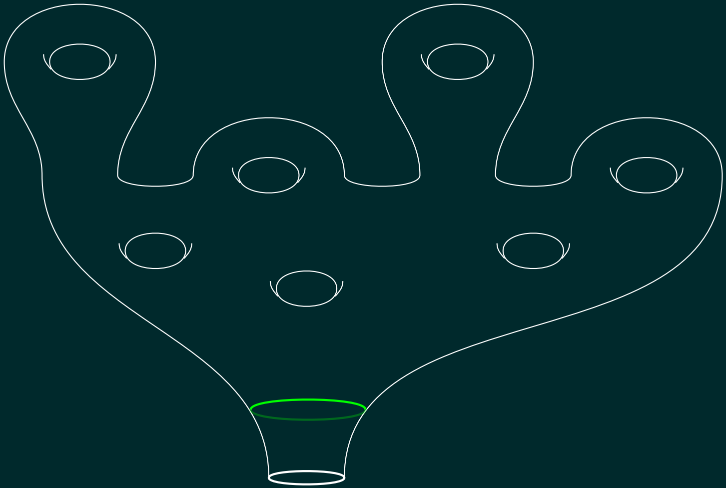

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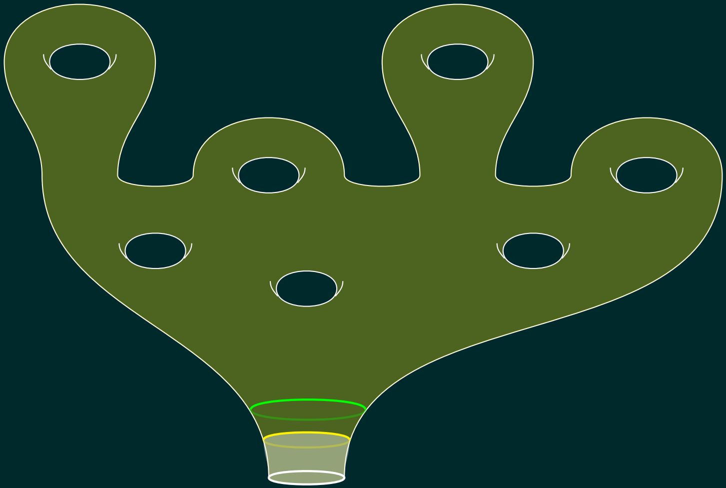


Cut locus of a mapping class

Let $\varphi \in \Gamma_{g,n}$ Then there is a unique isotopy class of an *oriented, unordered* multicurve $c_1, \ldots, c_h$ in $\mathcal{S}$ satisfying the following:

- $c_1, \ldots, c_h$ are disjoint simple closed curves, dividing $\Sigma_{g,n}$ into two regions W and Y, and are oriented as boundary curves of Y;
- each connected component of W touches $\partial\Sigma_{g,n}$;
- no connected component of Y is a disc;
- φ can be represented by a diffeomorphism $\Phi \colon \Sigma_{g,n} \to \Sigma_{g,n}$ fixing W pointwise;
- the isotopy class of $W \subset \Sigma_{g,n}$ is maximal among all isotopy classes of subsurfaces satisfying all the above conditions.

Definition

The isotopy class of multicurve $[c_1, \ldots, c_h]$ is called the *cut locus* of φ.
A mapping class $\varphi \in \Gamma_{g,n}$ is *∂-irreducible* if its cut locus is $[\partial\Sigma_{g,n}]$.

In fact, for a generic $\varphi \in \Gamma_{g,n}$ as above, the restriction of Φ to any component $P \subset Y$ gives a ∂-irreducible mapping class $\varphi_P \in \Gamma(P, \partial P)$.
Assume now $\varphi \in \Gamma_{g,1}$: then W is connected; instead Y may be disconnected, and connected components of Y may have more than 1 boundary curve!

Structure result for $Z(\varphi, \Gamma_{g,1})$, first part

Let $\varphi \in \Gamma_{g,1}$, and decompose $\Sigma_{g,1} = W \cup Y$ along the cut locus $c_1, \dots, c_h$.
Fix parametrisations $c_i \cong S^1$ which are compatible with orientation as ∂Y.
Represent φ by a diffeomorphism Φ fixing W pointwise, and let $\varphi_Y := [\Phi|_Y] \in \Gamma(Y, \partial Y)$.

The *extended mapping class group* $\Gamma(Y)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of Y that may permute the h components of ∂Y, but are compatible with their parametrisation.

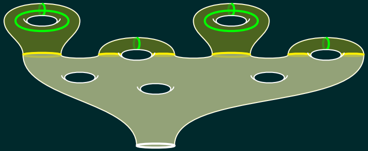
Similarly, the extended mapping class group $\Gamma^{\mathfrak{S}_h}(W)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of W that fix $\partial\Sigma_{g,1}$ pointwise, and may permute the other h components of ∂W, but are compatible with their parametrisation.

Both groups map to $\mathfrak{S}_h$. We have a gluing map

$$\hat{\varepsilon}\colon \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} \Gamma(Y) \to \Gamma_{g,1}.$$

Note that $\hat{\varepsilon}(\mathrm{Id}_W, \varphi_Y) = \varphi$. In fact $\hat{\varepsilon}$ restricts to

$$\varepsilon\colon \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} Z(\varphi_Y, \Gamma(Y)) \to Z(\varphi, \Gamma_{g,1}).$$



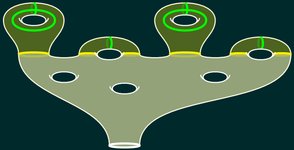
Proposition

The map ε is surjective and has kernel isomorphic to $\mathbb{Z}^h$, generated by the pairs $(D_{c_i}, D_{c_i}^{-1})$ for $1 \leqslant i \leqslant h$.

Recall that we have a central extension

$$\mathbb{Z}^h \longrightarrow \Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} Z(\varphi_Y, \Gamma(Y)) \xrightarrow{\varepsilon} Z(\varphi, \Gamma_{g,1}).$$

Recall that Y may be disconnected! Each component $P \subset Y$ is equipped with a mapping class $\varphi_P = [\Phi|_P] \in \Gamma(P, \partial P)$.
Two components $P, P' \subset Y$ are *similar* if there is $\Xi \colon P \to P'$ such that $(-)^{\Xi} \colon \Gamma(P, \partial P) \to \Gamma(P', \partial P')$ sends $\varphi_P \mapsto \varphi_{P'}$.



Decompose $Y = \coprod_{i=1}^{r} \coprod_{j=1}^{s_i} Y_{i,j}$, with $Y_{i,j}$ similar to $Y_{i',j'}$ iff $i = i'$. Let $Y_{i,j}$ be of type Σ_{g_i,n_i}, and let $\bar{\varphi}_i \in \Gamma_{g_i,n_i}$ correspond to $\varphi_{Y_{i,j}}$ for all j. We can further decompose

$$Z(\varphi_Y, \Gamma(Y)) \cong \prod_{i=1}^{r} \Big((Z(\bar{\varphi}_i, \Gamma(\Sigma_{g_i,n_i})))^{s_i} \rtimes \mathfrak{S}_{s_i} \Big).$$

Let $\mathfrak{H}_i \subset \mathfrak{S}_{n_i}$ be the image of $Z(\bar{\varphi}, \Gamma(\Sigma_{g_i,n_i}))$ along the natural map to $\mathfrak{S}_{n_i}$.
Then $\mathfrak{H} := \prod_{i=1}^{r} (\mathfrak{H}_i)^{s_i} \rtimes \mathfrak{S}_{s_i}$ is the subgroup of $\mathfrak{S}_h$ really used in the fibre product, and

$$\Gamma^{\mathfrak{S}_h}(W) \times^{\mathfrak{S}_h} Z(\varphi_Y, \Gamma(Y)) \cong \Gamma^{\mathfrak{H}}(W) \times^{\mathfrak{H}} \prod_{i=1}^{r} \Big((Z(\bar{\varphi}_i, \Gamma(\Sigma_{g_i,n_i})))^{s_i} \rtimes \mathfrak{S}_{s_i} \Big).$$

From groups to classifying spaces

Recall that we have a central extension, where $\mathfrak{H} := \prod_{i=1}^{r} (\mathfrak{H}_i)^{s_i} \rtimes \mathfrak{S}_{s_i} \subset \mathfrak{S}_h$:

$$\mathbb{Z}^h \longrightarrow \Gamma^{\mathfrak{H}}(W) \times^{\mathfrak{H}} \prod_{i=1}^{r} \left((Z(\bar{\varphi}_i, \Gamma(\Sigma_{g_i,n_i})))^{s_i} \rtimes \mathfrak{S}_{s_i} \right) \xrightarrow{\varepsilon} Z(\varphi, \Gamma_{g,1}).$$

Taking classifying spaces (and after some routine work) we get

$$B\Gamma(W, \partial W) \times_{T^h \rtimes \mathfrak{H}} \prod_{i=1}^{r} \left((BZ(\bar{\varphi}_i, \Gamma_{g_i,n_i}))^{s_i} \right) \simeq BZ(\varphi, \Gamma_{g,1}) \overset{\simeq}{\subset} \Lambda\mathfrak{M}_{g,1}.$$

- To describe $BZ(\varphi, \Gamma_{g,1})$ for $\varphi \in \Gamma_{g,1}$, we use as "bulding blocks" the spaces $BZ(\bar{\varphi}_i, \Gamma_{g_i,n_i}) \overset{\simeq}{\subset} \Lambda\mathfrak{M}_{g_i,n_i}$, corresponding to the ∂-irreducible mapping classes $\bar{\varphi}_i \in \Gamma_{g_i,n_i}$.
- These building blocks are assembled together using the space $B\Gamma(W, \partial W)$.
- Furthermore, we have some balancing by the group

$$T^h \rtimes \mathfrak{H} = \prod_{i=1}^{r} (T^{n_i} \rtimes \mathfrak{H}_i)^{s_i} \rtimes \mathfrak{S}_{s_i}.$$

Coloured operads and their algebras

For a set N, an N-coloured operad $\mathcal{O}$ consists of spaces of "operations" $\mathcal{O}\binom{k_1,\dots,k_r}{n}$ for all $r \geqslant 0$ and $k_1,\dots,k_r,n \in N$. We say that $k_1,\dots,k_r$ are the input colours, and n is the output colour. Composition of operations and permutation of inputs are only defined if they are compatible with the colours: for instance we have compositions

$$\mathcal{O}\binom{k_1,\dots,k_r}{n} \times \prod_{i=1}^{r} \mathcal{O}\binom{l_{i1},\dots,l_{is_i}}{k_i} \to \mathcal{O}\binom{l_{11},\dots,l_{rs_r}}{n}$$

An $\mathcal{O}$-algebra is a sequence of spaces $\mathbf{X} = (X_n)_{n\in N}$ with compatible multiplication maps

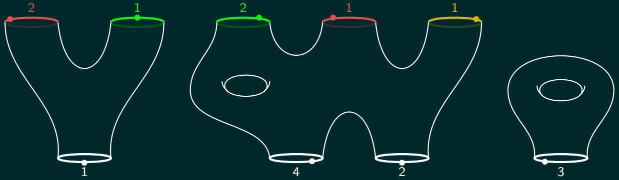
$$\mathcal{O}\binom{k_1,\dots,k_r}{n} \times \prod_{i=1}^{r} X_{k_i} \to X_n.$$

If $\mathcal{O}' \subset \mathcal{O}$ is a sub-N-coloured operad, every $\mathcal{O}$-algebra is a $\mathcal{O}'$-algebra. Viceversa, given a $\mathcal{O}'$-algebra $\mathbf{X} = (X_n)_n$, we can construct the relatively free $\mathcal{O}$-algebra $F^{\mathcal{O}}_{\mathcal{O}'}(\mathbf{X})$.

The coloured surface operad $\mathcal{M}$

For $N = \{1, 2, 3, \dots\}$ there is an N-coloured operad $\mathcal{M}$, whose restriction to colour-1 is Tillmann's surface operad. For generic $k_1, \dots, k_r, n \in N$, the space $\mathcal{M}\binom{k_1,\dots,k_r}{n}$ is the moduli space of compact Riemann surfaces W with the following additional structures:

- ∂W is partitioned as $\partial^{\mathrm{in}} W \sqcup \partial^{\mathrm{out}} W$, and each component of W touches $\partial^{\mathrm{out}} W$;
- $\partial^{\mathrm{out}} W$ is equipped with a diffeomorphism to $\{1, \dots, n\} \times S^1$;
- $\partial^{\mathrm{in}} W$ is equipped with a diffeomorphism to $\coprod_{i=1}^{r} \{1, \dots, k_i\} \times S^1$.



$\mathcal{M}$ contains an N-sequence of groups $\mathbf{R} = (R_n)_n$: the group $R_n := T^n \rtimes \mathfrak{S}_n$ embeds into $\mathcal{M}\binom{n}{n}$ as moduli of Riemann surfaces which are disjoint unions of cylinders.

$(z_1, \dots, z_n, \sigma) =$  $\in R_n \subset \mathcal{M}\binom{n}{n}$

In particular every $\mathcal{M}$-algebra is a $\mathbf{R}$-algebra. Viceversa, given a $\mathbf{R}$-algebra $\mathbf{X} = (X_n)_n$, i.e. a sequence of R_n-spaces X_n, we can form $F^{\mathcal{M}}_{\mathbf{R}}(\mathbf{X})$.

$\Lambda\mathfrak{M}_{*,1}$ as a relatively free algebra

For all $g \geqslant 0$ and $n \geqslant 1$ the group $R_n = T^n \rtimes \mathfrak{S}_n$ acts on the space $\mathfrak{M}_{g,n}$: given $\mathcal{S}$ with ordered and parametrised boundary components, we can reorder the labels $1, \dots, n$ of the boundary components, and rotate the parametrisations. R_n also acts Λ-wise on $\Lambda\mathfrak{M}_{g,n}$:

$$R_n \circlearrowleft \mathfrak{M}_{g,n}, \qquad R_n \circlearrowleft \Lambda\mathfrak{M}_{g,n}.$$

In fact $\mathfrak{M}_{g,n} \subset \mathcal{M}\binom{}{n}$, *and* $R_n \subset \mathcal{M}\binom{n}{n}$.

Recall that $\Lambda\mathfrak{M}_{g,n} = \coprod_{[\varphi]\in\mathrm{Conj}(\Gamma_{g,n})} \Lambda\mathfrak{M}_{g,n}(\varphi)$. Put

$$\mathfrak{C}_{g,n} := \coprod_{\substack{[\varphi]\in\mathrm{Conj}(\Gamma_{g,n}) \\ \varphi \text{ is } \partial\text{-irreducible}}} \Lambda\mathfrak{M}_{g,n}(\varphi) \subset \Lambda\mathfrak{M}_{g,n}.$$

Then R_n acts on $\mathfrak{C}_n := \coprod_{g\geqslant 0} \mathfrak{C}_{g,n}$, so $\mathfrak{C} := (\mathfrak{C}_n)_{n\in N}$ is a $\mathbf{R}$-algebra.

Proposition (improving on the naive conjecture)

$\Lambda\mathfrak{M}_{*,1}$ is the colour-1 part of $F^{\mathcal{M}}_{\mathbf{R}}(\mathfrak{C})$.

BBPTY for coloured operads with homological stability

There is a notion of *N-coloured OHS* for an arbitrary set N. Let $\mathcal{O}$ be an N-coloured OHS containing a sequence of groups $\mathbf{G} = (G_n)_n$ as suboperad (of 1-ary operations). Let $\mathbf{X} = (X_n)_n$ be a $\mathbf{G}$-algebra. Let $n \in N$; then $F^{\mathcal{O}}_{\mathbf{G}}(\mathbf{X})_n$ is in particular a topological monoid.

Proposition

There is an equivalence of loop spaces

$$\Omega B F^{\mathcal{O}}_{\mathbf{G}}(\mathbf{X})_n \simeq \Omega B \mathcal{O}\binom{}{n} \times \Omega^\infty \Sigma^\infty_+ \left(\coprod_{k \in N} X_k // G_k \right).$$

Let now $N = \{1, 2, 3, \dots\}$: then $\mathcal{M}$ is an N-coloured OHS containing $\mathbf{R}$ as suboperad. Recall that $\Lambda\mathfrak{M}_{*,1}$ is the colour-1 part of $F^{\mathcal{M}}_{\mathbf{R}}(\mathfrak{C})$, where $\mathfrak{C}_n$ is the R_n-space

$$\mathfrak{C}_n = \coprod_{g \geqslant 0} \coprod_{\substack{[\varphi] \in \mathrm{Conj}(\Gamma_{g,n}) \\ \partial-\text{irreducible}}} \Lambda\mathfrak{M}_{g,n}(\varphi).$$

We then have an equivalence of loop spaces

$$\Omega B \Lambda\mathfrak{M}_{*,1} \simeq \Omega B \mathcal{M}\binom{}{1} \times \Omega^\infty \Sigma^\infty_+ \left(\coprod_{n \geqslant 1} \mathfrak{C}_n // R_n \right) \simeq \Omega^\infty \mathrm{MTSO}(2) \times \Omega^\infty \Sigma^\infty_+ \mathfrak{X}.$$