Anomaly Constraints in Spontaneous Symmetry Broken Phases

A Long Exact Sequence from Smith Homomorphisms

Cameron Krulewski

Mathematics PhD Student
Massachusetts Institute of Technology

Joint with A. Debray, S. Devalapurkar, Y. L. Liu, N. Pacheco-Tallaj, and R. Thorngren

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Overview

- Smith homomorphisms are maps between bordism groups that change dimension and tangential structure.
- They induce maps on topological field theories corresponding to spontaneous symmetry breaking processes. [HKT20]
- We study the maps of spectra inducing Smith homomorphisms and interpret the physics of the corresponding long exact sequence.

Outline

Idea

- Fix a dimension d, a stable tangential structure $\eta\colon X \to BO$, and twisting data $\rho\colon X \to BO(k)$.
  - ex. $X = B\mathrm{Spin} \times B\mathbb{Z}/2$, $\eta\colon B\mathrm{Spin} \times B\mathbb{Z}/2 \to B\mathrm{Spin} \hookrightarrow BO$, $\rho = \sigma$ the sign representation

Idea

A *Smith homomorphism* is a map on bordism groups

$$\Omega_d^{\eta}(X) \xrightarrow{sm} \Omega_{d-k}^{\eta-\rho}(X).$$

- $\Omega_d^{\eta}(X)$ is the group of bordism classes of closed d-manifolds with (X, η) structure
- $[M, \omega] \in \Omega_d^{\eta}(X)$ consists of M a closed d-manifold with a lift

$$\begin{array}{ccc} & & X \\ & \overset{\omega}{\nearrow} & \downarrow \eta \\ M & \xrightarrow{TM} & BO \end{array}$$

Examples

- classical ex: $\tilde{\Omega}^{\eta}_{d}(S \times B\mathbb{Z}/2) \xrightarrow{\cong} \Omega^{\eta-\sigma}_{d-1}(S \times B\mathbb{Z}/2)$, S any structure [Gil18], [CF64]
- "$\mathbb{Z}/2$ ex.": $X = B\mathrm{Spin} \times B\mathbb{Z}/2$, $\rho = \sigma$ the sign representation, $k = 1$
  - $\Omega^{\eta}_{d}(B\mathrm{Spin} \times B\mathbb{Z}/2) \to \Omega^{\eta-\sigma}_{d-1}(B\mathrm{Spin} \times B\mathbb{Z}/2)$
  - $\approx \Omega^{\mathrm{Spin}\times\mathbb{Z}/2}_{d} \to \Omega^{\mathrm{Pin}^-}_{d-1}$
  - iterate: $\Omega^{\eta-\sigma}_{d-1}(B\mathrm{Spin} \times B\mathbb{Z}/2) \to \Omega^{\eta-2\sigma}_{d-2}(B\mathrm{Spin} \times B\mathbb{Z}/2)$
  - $\approx \Omega^{\mathrm{Pin}^-}_{d-1} \to \Omega^{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4}_{d-2}$
  - periodic family: $\Omega^{\mathrm{Spin}\times\mathbb{Z}/2}_{d} \to \Omega^{\mathrm{Pin}^-}_{d-1} \to \Omega^{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4}_{d-2} \to \Omega^{\mathrm{Pin}^+}_{d-3} \to \Omega^{\mathrm{Spin}\times\mathbb{Z}/2}_{d-4} \to \dots$
  - [Kap+15; WWZ20; HKT20; TY19]
- "$U(1)$ ex.": $X = B\mathrm{Spin} \times BU(1)$, $\rho = \gamma$ tautological bundle over $BU(1)$, $k = 2$
  - $\Omega^{\eta}_{d}(B\mathrm{Spin} \times BU(1)) \to \Omega^{\eta-\gamma}_{d-2}(B\mathrm{Spin} \times BU(1))$
  - $\approx \Omega^{\mathrm{Spin}}_{d}(BU(1)) \to \Omega^{\mathrm{Spin}^c}_{d-2}$
  - periodic family $\Omega^{\mathrm{Spin}}_{d}(BU(1)) \to \Omega^{\mathrm{Spin}^c}_{d-2} \to \Omega^{\mathrm{Spin}}_{d-4}(BU(1)) \to \dots$

Identifying Tangential Structures

- "$\mathbb{Z}/2$ ex.": $\Omega_d^{\mathrm{Spin}\times\mathbb{Z}/2} \to \Omega_{d-1}^{\mathrm{Pin}^-} \to \Omega_{d-2}^{\mathrm{Spin}^c} \to \Omega_{d-3}^{\mathrm{Pin}^+}$ [BC18; Sto88; KT90; Pet68]
  - shearing: $\mathrm{Pin}^\pm$ and $\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4$ can be realized as twisted spin structures
  - e.g. pin^+-structure $\leftrightarrow$ $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structure:
    - a pin^+ structure on E is a trivialization $w_2(E) = 0$
    - equivalently, it is a spin structure on $E \oplus 3\mathrm{Det}(E)$ (i.e., $w_2(E \oplus 3\mathrm{Det}(E)) = 0 = w_1(E \oplus 3\mathrm{Det}(E))$)
    - a $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structure on $E \to M$ is a a map $f\colon M \to B\mathbb{Z}/2$ and a spin structure on $E \oplus 3f^*(\sigma)$
- "$U(1)$ ex.": $\Omega_d^{\mathrm{Spin}} \to \Omega_{d-2}^{\mathrm{Spin}^c}$
  - shearing: a (Spin, γ) structure is equivalent to a Spin^c structure

Questions

- What's the general construction for these maps?
- When are Smith homomorphisms isomorphisms? When are they injective/surjective?
- Why do they often form periodic families?

Construction for Fixed $(d, \eta\colon X \to BO, \rho\colon X \to BO(k))$

$$\begin{array}{ccc} \Omega_d^\eta(X) & \xrightarrow{\quad sm \quad} & \Omega_{d-k}^{\eta-\rho}(X) \\ \Cup & & \Cup \\ [M,\omega] & \longmapsto & [N,\omega'] \end{array}$$

- $[M,\omega] \in \Omega_d^\eta(X)$ is a closed manifold with a lift ω of its stable tangential structure

$$\begin{array}{ccc} & & X \\ & \overset{\omega}{\nearrow} & \downarrow \eta \\ M & \xrightarrow{TM} & BO \end{array}$$

- $\rho \rightsquigarrow$ k-dimensional vector bundle $V \to X$
- generic section $s\colon M \to V$ is transverse to s_0
- define submanifold N by pullback

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & \lrcorner & \downarrow s_0 \\ M & \xrightarrow{s} & V \end{array} \qquad \text{i.e. } N = M_s \cap M_{s_0}$$

- tangential structure on N?

$$\begin{array}{ccccc} & & & \overset{\omega|_N}{\dashrightarrow} & X \\ & & & \overset{\omega}{\nearrow} & \downarrow \eta \\ N & \hookrightarrow & M & \xrightarrow{TM} & BO \end{array} \quad \rightsquigarrow \quad TM|_N$$

Tangential Structure on Submanifold N

Claim

Stably, $TN \cong TM|_N - V|_N$. Therefore, $[N, \omega'] \in \Omega^{\eta-\rho}_{d-k}(X)$.

Proof.

- By transversality, $TM_{s_0}|_N \oplus TM_s|_N \twoheadrightarrow V$.
- By construction, there is exact sequence of vector bundles

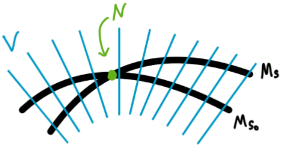
$$0 \to TM_{s_0} \to TV|_{M_{s_0}} \to V \to 0.$$

- Take $\cap TM_s$ to get

$$0 \to TN \to TM_s|_N \to V \to 0.$$

□

Definition of N



Tangential Structure

$$N \hookrightarrow M \xrightarrow{V} BO(k), \quad \rho: X \to BO(k), \quad \omega: M \dashrightarrow X, \quad \omega': N \dashrightarrow X$$

Map of Spectra

Theorem (Pontryagin-Thom)

Let $\eta : X \to BO$ be a tangential structure and let $-\eta$ be its inverse. There is an isomorphism of groups $\Omega_d^\eta(X) \cong \pi_d(X^{-\eta})$.

Proposition

The Smith homomorphism $\Omega_d^\eta(X) \to \Omega_{d-k}^{\eta-\rho}(X)$ is induced by a map of spectra

$$X^{-\eta} \xrightarrow{sm} X^{-\eta+\rho}.$$

- take $-\eta$ to be an actual bundle μ for convenience
- define a map of vector bundles $\mu \to \mu + \rho$ by taking the zero section of ρ
- $\rightsquigarrow$ a map $S(\mu) \to S(\mu + \rho)$ of sphere bundles over X
- $\rightsquigarrow$ a map on Thom spaces $\mathrm{Th}(X; \mu) \to \mathrm{Th}(X; \mu + \rho)$
- note that $X^{-\eta+\rho} \simeq \Sigma^k X^{-\eta+\rho-k}$

Example Maps of Spectra

Proposition

The Smith homomorphism $\Omega_d^{\eta}(X) \to \Omega_{d-k}^{\eta-\rho}(X)$ is induced by a map of spectra

$$X^{-\eta} \xrightarrow{sm} X^{-\eta+\rho}.$$

- "$\mathbb{Z}/2$": $\Omega_d^{\mathrm{Spin}\times\mathbb{Z}/2} \to \Omega_{d-1}^{\mathrm{Pin}^-}$ is induced by
  - $MT\mathrm{Spin} \otimes B\mathbb{Z}/2 \xrightarrow{sm} MT\mathrm{Spin} \otimes (B\mathbb{Z}/2)^{\sigma} \simeq \Sigma MT\mathrm{Spin} \otimes (B\mathbb{Z}/2)^{\sigma-1} \simeq \Sigma MT\mathrm{Pin}^-$
- "$U(1)$": $\Omega_d^{\mathrm{Spin}} \to \Omega_{d-2}^{\mathrm{Spin}^c}$ is induced by
  - $MT\mathrm{Spin} \otimes BU(1) \xrightarrow{sm} MT\mathrm{Spin} \otimes BU(1)^{\gamma} \simeq \Sigma^2 MT(\mathrm{Spin} \times_{\mathbb{Z}/2} U(1)) \simeq \Sigma^2 MT\mathrm{Spin}^c$

Questions

- What's the general construction for these maps? ✓
- When are Smith homomorphisms isomorphisms? When are they injective/surjective?
- Why do they often form periodic families?

Cofiber of Smith Map

Proposition

Let $p\colon S(\rho) \to X$ be the projection. There is a cofiber sequence of pointed spaces

$$\mathrm{Th}(X;\mu) \xrightarrow{sm} \mathrm{Th}(X;\mu+\rho) \longrightarrow \Sigma\mathrm{Th}(S(\rho);p^*(\mu)).$$

Proof.

- There's a pushout square

$$\begin{array}{ccc} S(\mu)\times_X S(\rho) & \longrightarrow & S(\rho) \\ \downarrow & & \downarrow \\ S(\mu) & \longrightarrow & S(\mu+\rho) \end{array}$$

so the cofibers of each row are the same.

- identify $S(\mu)\times_X S(\rho) \simeq S(p^*(\mu))$
- $\implies$ $\mathrm{cofib}(S(p^*(\mu)) \to S(\rho)) \simeq \mathrm{Th}(S(\rho);p^*(\mu))$

□

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Proof (cont'd).

There's a commutative diagram

$$\begin{array}{ccccc}
S(\mu) & \longrightarrow & X & \longrightarrow & \mathrm{Th}(X;\mu) \\
\downarrow & & \| & & \downarrow{\scriptstyle sm} \\
S(\mu+\rho) & \longrightarrow & X & \longrightarrow & \mathrm{Th}(X;\mu+\rho) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Th}(S(\rho);p^*(\mu)) & \longrightarrow & * & \longrightarrow & \mathrm{cofib}(sm)
\end{array}$$

- each row and column is a cofiber sequence
- from the last row, see $\mathrm{cofib}(sm) \simeq \Sigma\mathrm{Th}(S(\rho);p^*(\mu))$

□

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Corollary

There is a cofiber sequence of spectra

$$S(\rho)^{p^*(\mu)} \to X^{\mu} \xrightarrow{sm} X^{\mu+\rho}.$$

- $MT\mathrm{Spin} \to MT\mathrm{Spin} \otimes B\mathbb{Z}/2 \xrightarrow{sm} \Sigma MT\mathrm{Pin}^-$
- $MT\mathrm{Spin} \to MT\mathrm{Pin}^- \xrightarrow{sm} \Sigma MT\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4$
- $MT\mathrm{Spin} \to MT\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4 \xrightarrow{sm} \Sigma MT\mathrm{Pin}^+$
- $MT\mathrm{Spin} \to MT\mathrm{Pin}^+ \xrightarrow{sm} \Sigma MT\mathrm{Spin} \otimes B\mathbb{Z}/2$

Cofiber of Smith Map

Proposition

Let $p\colon S(\rho) \to X$ be the projection. There is a cofiber sequence of pointed spaces

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Corollary

There is a cofiber sequence of spectra

$$S(\rho)^{p^*(\mu)} \to X^{\mu} \xrightarrow{sm} X^{\mu+\rho}.$$

- $MT\mathrm{Spin} \to MT\mathrm{Spin} \otimes BU(1) \xrightarrow{sm} \Sigma^2 MT\mathrm{Spin}^c$
- $MT\mathrm{Spin} \to MT\mathrm{Spin}^c \xrightarrow{sm} \Sigma^2 MT\mathrm{Spin} \otimes BU(1)$

Cofiber of Smith Map

Proposition

Let $p\colon S(\rho) \to X$ be the projection. There is a cofiber sequence of pointed spaces

$$\mathrm{Th}(X;\mu) \xrightarrow{sm} \mathrm{Th}(X;\mu+\rho) \longrightarrow \Sigma\mathrm{Th}(S(\rho);p^*(\mu)).$$

Corollary

There is a cofiber sequence of spectra

$$S(\rho)^{p^*(\mu)} \to X^{\mu} \xrightarrow{sm} X^{\mu+\rho}.$$

- consider subfamilies of the $\mathbb{Z}/2$ family with $\rho = 2\sigma$
- $MT\mathrm{Spin} \otimes \Sigma_+^{\infty-1}\mathbb{R}P^2 \to MT\mathrm{Pin}^- \xrightarrow{sm} \Sigma^2 MT\mathrm{Pin}^+$ [KT90]

Questions

- What's the general construction for these maps? ✓
- When are Smith homomorphisms isomorphisms? When are they injective/surjective? ✓
- Why do they often form periodic families?

Periodic Families

- periodic Smith families occur when $X^{-\eta+n\rho} \simeq \Sigma^{kn} X^{-\eta}$ for some n
- take $X = B\mathrm{Spin} \times X_0$, $\eta\colon X = B\mathrm{Spin} \times X_0 \to B\mathrm{Spin} \hookrightarrow BO$, ρ nontrivial on X_0

Fact

Consider twisting data $\rho\colon X_0 \to BO(k)$. There is an isomorphism (over $M\mathrm{Spin}$)

$$X_0^{n\rho} \otimes M\mathrm{Spin} \simeq \Sigma^{nk} X_0 \otimes M\mathrm{Spin}$$

if and only if $n\rho$ has a spin structure.

- "$\mathbb{Z}/2$ ex.": $\mathrm{Spin} \times \mathbb{Z}/2 \rightsquigarrow \mathrm{Pin}^- \rightsquigarrow \mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4 \rightsquigarrow \mathrm{Pin}^+ \rightsquigarrow \mathrm{Spin} \times \mathbb{Z}/2 \rightsquigarrow \dots$
  - for any real vector bundle E, $2E$ is always oriented and $4E$ is always spin
  - 4σ spin $\implies$ $M\mathrm{Spin} \otimes (B\mathbb{Z}/2)^{4\sigma} \simeq \Sigma^4 M\mathrm{Spin} \otimes B\mathbb{Z}/2$
- "$U(1)$ ex.": $\mathrm{Spin} \rightsquigarrow \mathrm{Spin}^c \rightsquigarrow \mathrm{Spin} \rightsquigarrow \dots$
  - for any complex vector bundle E, E is always oriented and $2E$ is always spin

Questions

- What's the general construction for these maps? ✓
- When are Smith homomorphisms isomorphisms? When are they injective/surjective? ✓
- Why do they often form periodic families? ✓

Induced Map of Invertible Topological Field Theories

- a d-dimensional TFT is a symmetric monoidal functor $F\colon \mathrm{Bord}_d^{(X,\eta)} \to \mathcal{C} = I\mathbb{Z}$

Ansatz [FH21], [Gal+10], [Ngu17]

There is a bijection

$$\left\{\begin{array}{l}\text{iso classes of cts reflection positive}\\ \text{invertible } d\text{-dim'l extended TFTs}\\ \text{with symmetry type } (X,\eta)\end{array}\right\} \simeq [X^{-\eta}, \Sigma^{d+1} I\mathbb{Z}] \simeq I\mathbb{Z}^{d+1}(X^{-\eta})$$

- let $sm\colon X^{-\eta} \to X^{-\eta+\rho}$ be a Smith map
- $\rightsquigarrow$ induced map $sm\colon |\mathrm{Bord}_d^{(X,\eta)}| \xrightarrow{sm} |\mathrm{Bord}_{d-k}^{(X,\eta-\rho)}|$
- fix a *defect theory* associated to an iTFT $F\colon \mathrm{Bord}_{d-k}^{(X,\eta-\rho)} \to I\mathbb{Z}$
- compose $|\mathrm{Bord}_d^{(X,\eta)}| \xrightarrow{sm} |\mathrm{Bord}_{d-k}^{(X,\eta-\rho)}| \xrightarrow{F} I\mathbb{Z}$ to get an iTFT for the *bulk theory*

Anomalies of Field Theories

Ansatz [FH21], [Gal+10], [Ngu17]

There is a bijection

$$\left\{\begin{array}{l}\text{iso classes of cts reflection positive}\\ \text{invertible } d\text{-dim'l extended TFTs}\\ \text{with symmetry type } (X,\eta)\end{array}\right\} \simeq [X^{-\eta}, \Sigma^{d+1} I\mathbb{Z}] \simeq I\mathbb{Z}^{d+1}(X^{-\eta})$$

- an *anomaly* of a d-dimensional anomalous field theory is a $d+1$-dimensional invertible topological* field theory
- anomalies are useful invariants of field theories (e.g. invariant under RG flow)
- *sm* induces a map $I\mathbb{Z}^{d+1}(X^{\eta}) \to I\mathbb{Z}^{d+1-k}(X^{-\eta+\rho})$
- we perform *'t Hooft anomaly matching* under *spontaneous symmetry breaking*

Anomaly Matching Hypothesis

Fix

- tangential structure $\eta\colon X \to BO$
- a d-dim'l *bulk field theory* with (X, η) structure and anomaly class $\alpha \in I\mathbb{Z}^{d+1}(X^{-\eta})$
- a k-dimensional symmetry-breaking order parameter ϕ transforming in the representation $\rho\colon X \to BO(k)$
- *and assume the IR limit of the theory is gapped.*

Then,

- a *twisted boundary condition construction* can produce a defect with excitations localized at $\langle\phi\rangle = 0$ (corresponding to $M_s \cap M_{s_0}$)
- the *defect field theory* has anomaly class $\beta \in I\mathbb{Z}^{d+1-k}(X^{-\eta+\rho})$
- such that if sm^* is the induced map, $\boxed{\alpha = sm^*(\beta)}$.

Application: Anomaly Matching in Symmetry Broken Phase with $U(1)$ Symmetry

- consider the Smith map $sm\colon M\mathrm{Spin} \otimes BU(1) \to \Sigma^2 M\mathrm{Spin}^c$
- $\rightsquigarrow$ map of field theories $I\mathbb{Z}^{d+1-2}(M\mathrm{Spin}^c) \to I\mathbb{Z}^{d+1}(M\mathrm{Spin} \otimes BU(1))$
- $\mathrm{cofib}(M\mathrm{Spin} \otimes BU(1) \to \Sigma^2 M\mathrm{Spin}^c) \simeq M\mathrm{Spin}$
- $\rightsquigarrow$ LES of field theories

Long Exact Sequence for “$U(1)$ Example”

	$I\mathbb{Z}^{*-2}(\mathrm{Spin}^c)$		$I\mathbb{Z}^*(M\mathrm{Spin} \otimes BU(1))$		$I\mathbb{Z}^*(M\mathrm{Spin})$
-1	0		$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}$
0	0		0		0
1	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/2$
2	0		$\mathbb{Z}/2$	$\longrightarrow$	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}^2$	$\longrightarrow$	$\mathbb{Z}$
4	0		0		0
5	$\mathbb{Z}^2$	$\longrightarrow$	$\mathbb{Z}^2$		0

Application: Anomaly Matching in Symmetry Broken Phase with $U(1)$ Symmetry

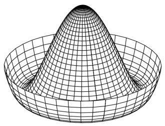
- consider the Smith map $sm\colon M\mathrm{Spin} \otimes BU(1) \to \Sigma^2 M\mathrm{Spin}^c$
- $\rightsquigarrow$ map of field theories $I\mathbb{Z}^{d+1-2}(M\mathrm{Spin}^c) \to I\mathbb{Z}^{d+1}(M\mathrm{Spin} \otimes BU(1))$
- $\mathrm{cofib}(M\mathrm{Spin} \otimes BU(1) \to \Sigma^2 M\mathrm{Spin}^c) \simeq M\mathrm{Spin}$
- $\rightsquigarrow$ LES of field theories
- focus on $I\mathbb{Z}^3(M\mathrm{Spin}^c) \xrightarrow{\cong} I\mathbb{Z}^5(M\mathrm{Spin} \otimes BU(1))$
- **hypothesis**: the bulk and defect anomalies match

Long Exact Sequence for "$U(1)$ Example"

	$I\mathbb{Z}^{*-2}(\mathrm{Spin}^c)$		$I\mathbb{Z}^*(M\mathrm{Spin} \otimes BU(1))$		$I\mathbb{Z}^*(M\mathrm{Spin})$
-1	0		$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}$
0	0		0		0
1	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/2$
2	0		$\mathbb{Z}/2$	$\longrightarrow$	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}^2$	$\longrightarrow$	$\mathbb{Z}$
4	0		0		0
5	$\mathbb{Z}^2$	$\longrightarrow$	$\mathbb{Z}^2$		0

Application: Anomaly Matching in Symmetry Broken Phase with $U(1)$ Symmetry

- $3+1d$ bulk theory Dirac Lagrangian: $\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi$ with chiral $U(1)_L$ symmetry taking $\psi = \binom{\psi_L}{\psi_R} \mapsto \binom{e^{i\theta}\psi_L}{\psi_R}$
- add a "mass term" with ϕ transforming in the charge 1 representation of $U(1)$: $\mathcal{L}' = \bar{\psi}\partial\!\!\!/\psi + \phi\bar{\psi}\psi + \partial_\mu\phi\partial^\mu\phi$
- choose a $U(1)$-symmetric potential V: $V(\phi) = -m^2|\phi|^2 + \lambda^4|\phi|^4$
- impose *twisted boundary conditions* on ϕ: $\phi(r,\theta) = \phi_0(r)e^{i\theta}$, $(r,\theta) \leftrightarrow (x^2, x^3)$
- solve Dirac equation to find the theory on the defect, which is localized to $(x^2, x^3) = 0$ and is thus a $1+1d$ theory
- we compute the anomalies and the induced Smith map rationally and observe that they match



Application: Obstruction to Electroweak Symmetry Breaking

Smith map:

- take $X = B\mathrm{Spin} \times BSU(N-1)$ and ρ the tautological bundle over $BSU(N-1)$
- $\rightsquigarrow sm\colon M\mathrm{Spin} \otimes BSU(N-1) \to M\mathrm{Spin} \otimes BSU(N)$
- $\mathrm{cofib}(sm) \simeq M\mathrm{Spin} \otimes BSU(N-1)$

physical implications:

- consider a theory with $SU(N)$ symmetry
- can you produce a gapped theory by breaking this symmetry with ρ the tautological bundle?
- no, not in general, since $I\mathbb{Z}(M\mathrm{Spin} \otimes BSU(N-1))$ is nontrivial

Long Exact Sequence in Symmetry Breaking

- Joint with Arun Debray, Sanath Devalapurkar, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren
- See forthcoming work for interpretation of not only the Smith map and its cofiber but also the connecting map in the LES:

$$\cdots \to \Omega^D_{G,s}(S(\rho)) \xrightarrow{\text{connecting}} \Omega^{D+1-k}_{G,s} \xrightarrow{\text{Smith}} \Omega^{D+1}_{G,s} \xrightarrow{\text{cofiber}} \Omega^{D+1}_{G,s}(S(\rho)) \to \cdots$$

$$\Omega^{D+1-k}_{G,s} \xrightarrow[\simeq]{\text{Thom}} \Omega^{D+1}_{G,s}(D(\rho), S(\rho)) \xrightarrow{\delta} \Omega^{D+1}_{G,s}$$

- And further applications: QCD, DQCP, spin $\mathbb{C}P^1$ model, 10-fold way

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