Complex cobordism spectrum via algebraic varieties

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14 October 2022

Motivic homotopy theory is a mix of algebraic geometry and homotopy theory. Motivic counterparts of sets are Nisnevich sheaves of sets defined on the smooth algebraic varieties Sm/k over a field k. Sm/k is equipped with Nisnevich topology which is between Zariski topology and etale topology.

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The category of motivic spaces $\mathcal{M}$ has a motivic model structure. It is Bousfield localization of the local model structure on $\mathcal{M}$ with respect to the family

$$\{pr_X : X \times \mathbf{A}^1 \to X \mid X \in Sm/k\}.$$

The same applies to pointed motivic spaces $\mathcal{M}_\bullet$.

There are two circles in motivic homotopy theory. One circle is given by the usual simplicial circle S^1. The second circle, denoted by $\mathbf{G}$, is the mapping cone of the embedding $1 : pt \hookrightarrow \mathbf{G}_m$, where $\mathbf{G}_m := \mathrm{Spec}(k[t^{\pm}])$.

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We then stabilise the motivic model structure on $\mathcal{M}_\bullet$ with respect to S^1 and $\mathbf{G}$ arriving at the stable motivic model structure $Sp_{S^1,\mathbf{G}}(k)$ of "$(S^1, \mathbf{G})$-bispectra". Its homotopy theory is denoted by $SH(k)$. $SH(k)$ is called the *stable motivic homotopy category of k*.

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A serious advantage of $SH(k)$ over the classical SH in topology is that (pre)sheaves of stable homotopy groups on motivic bispectra can have various "correspondences". In practice all known types of correspondences form categories whose objects are those of Sm/k but whose morphisms are given by tricky geometric data.

If the base field k is $\mathbf{C}$, there is a natural realization functor

$$Re : SH(k) \to SH$$

Re is an extension of the functor

$$An : Sm_{\mathbf{C}} \to \mathbf{Top}$$

sending X to $X^{an} := X(\mathbf{C})$ with the classical topology.

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Such computations are possible due to the machinery of framed correspondences of Voevodsky and framed motives (in the sense of G.–Panin).

Implicit functions vs framings

Theorem (Implicit Function Theorem)

Let $F(z,w) = (F_1(z,w), \dots, F_m(z,w))$ *be complex polynomials in* $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_m)$ *and* $F(z^0, 0) = 0$ *for* $(z^0, 0) \in \mathbf{C}^n \times \mathbf{C}^m$*. Suppose*

$$\det \left(\frac{\partial F}{\partial w} \right) (z^0, 0) \neq 0.$$

Then the equations $F(z,w) = 0$ *have a uniquely determined holomorphic solution* $w = f(z) = (f_1(z), \dots, f_m(z))$ *in a neighbourhood of* z^0 *such that* $f(z^0)$*=0.*

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The theorem yields a triple $\Phi = (Z, U, f)$, where $Z := \{z^0\}$, U is a neighbourhood of Z, $f = (f_1, \dots, f_m) : U \to \mathbf{C}^m$ is a holomorphic map on U. If, moreover, $Z = f^{-1}(0)$ then we call the triple a *holomorphic framed correspondence*.

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The triple Φ gives a morphism of pointed motivic spaces

$$\Phi : \mathbf{P}^{\wedge n} \to T^m,$$

where $\mathbf{P}^{\wedge n} = (\mathbf{P}^1_{\mathbf{C}}, \infty) \wedge \overset{n}{\cdots} \wedge (\mathbf{P}^1_{\mathbf{C}}, \infty)$, $T^m = \mathbf{A}^m_{\mathbf{C}} / (\mathbf{A}^m_{\mathbf{C}} - 0)$.

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We likewise define triples $\Phi = (Z, U, f)$ with Z having finitely many points. Φ is *equivalent to* $\Phi' = (Z', U', f')$ if $Z = Z'$ and there is a smaller neighbourhood U'' of Z such that $f|_U = f'|_{U'}$. Then there is a bijective correspondence between equivalence classes of such triples and morphisms of pointed motivic spaces $\Phi : \mathbf{P}^{\wedge n} \to T^m$. This bijection is a version of Voevodsky's lemma in terms of holomorphic functions.

If $m = n$ the set of equivalence classes of triples $\Phi = (Z, U, f)$ is denoted by $Fr_n(\mathrm{pt}, \mathrm{pt})$. There are natural pairings

$$Fr_n(\mathrm{pt}, \mathrm{pt}) \times Fr_s(\mathrm{pt}, \mathrm{pt}) \to Fr_{n+s}(\mathrm{pt}, \mathrm{pt})$$

making the set $Fr_*(\mathbf{C}) := \bigsqcup_{n \geq 0} Fr_n(\mathrm{pt}, \mathrm{pt})$ a monoid in pointed simplicial sets. There is a natural stabilization map $\sigma : Fr_n(\mathrm{pt}, \mathrm{pt}) \to Fr_{n+1}(\mathrm{pt}, \mathrm{pt})$ sending (Z, U, f) to $(Z \times 0, U \times \mathbf{C}, f \circ pr_U)$. After stabilising one gets a pointed set $Fr(\mathrm{pt}, \mathrm{pt}) = \mathrm{colim}_n Fr_n(\mathrm{pt}, \mathrm{pt})$.

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So far we have dealt with classical complex analysis in several variables. The monoid $Fr_*(\mathbf{C})$ can be recovered from Voevodsky's framed correspondences defined for smooth k-schemes. The implicit functions $f = (f_1, \ldots, f_m) : U \to \mathbf{C}^m$ play the role of "framings" in the sense of Voevodsky.

Voevodsky's framed correspondences

Definition (Voevodsky (2001))

For k-smooth schemes $X, Y \in Sm/k$ and $n \geq 0$ an *explicit framed correspondence* Φ of level n consists of the following data:

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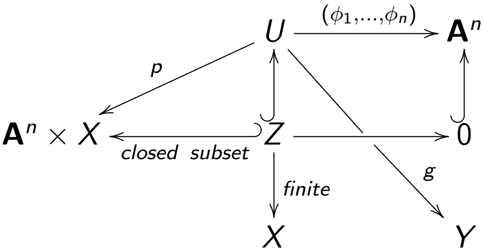
(4) a morphism $g : U \to Y$.

The subset Z will be referred to as the *support* of the correspondence. We shall also write triples $\Phi = (U, \phi, g)$ or quadruples $\Phi = (Z, U, \phi, g)$ to denote explicit framed correspondences.

Voevodsky's framed correspondences

A framed correspondence can be depicted as follows:



Two explicit framed correspondences Φ and Φ' of level n are said to be *equivalent* if they have the same support and there exists an etale neighborhood V of Z in $U \times_{\mathbf{A}^n_X} U'$ such that on V, the morphism $g \circ pr$ agrees with $g' \circ pr'$ and $\phi \circ pr$ agrees with $\phi' \circ pr'$. A *framed correspondence of level* n is an equivalence class of explicit framed correspondences of level n.

Denote the set of framed correspondences of level n by $Fr_n(X, Y)$.

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Lemma (Voevodsky)

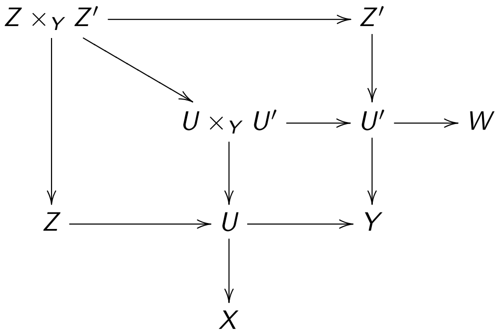
For any $X, Y \in Sm/k$ and any $n \geq 0$ there are natural isomorphisms

$$\begin{aligned} Fr_n(X, Y) &= \mathrm{Hom}_{Shv_\bullet^{nis}(Sm/k)}(X_+ \wedge (\mathbf{P}^1, \infty)^n, Y_+ \wedge (\mathbf{A}^1/(\mathbf{A}^1 - 0))^n) \\ &= \mathrm{Hom}_{Shv_\bullet^{nis}(Sm/k)}(X_+ \wedge (\mathbf{P}^1, \infty)^n, Y_+ \wedge T^n). \end{aligned}$$

We can compose framed correspondences:

$$Fr_n(X,Y) \times Fr_m(Y,W) \to Fr_{n+m}(X,W).$$

The composition is encoded by the following diagram:



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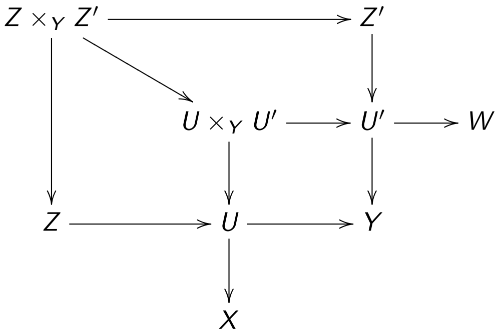

With this composition we form a category $Fr_*(k)$ whose objects are those of Sm/k and morphisms are given by

$$Fr_*(X,Y) = \sqcup_{n \geq 0} Fr_n(X,Y).$$

Voevodsky calls it the *category of framed correspondences*.

An important level 1 framed endomorphism $\sigma_X \in Fr_1(X, X)$ is given by $(X \times 0, X \times \mathbf{A}^1, pr_{\mathbf{A}^1}, pr_X)$. Using Voevodsky's lemma, σ_X corresponds to the canonical motivic equivalence $X_+ \wedge (\mathbf{P}^1, \infty) \to X_+ \wedge T$.

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We can stabilise in σ to get the set of *stable framed correspondences* $Fr(X, Y)$. The standard cosimplicial affine scheme is defined by

$$n \mapsto \Delta^n_k := \mathrm{Spec}(k[x_0, \dots, x_n]/(x_0 + \cdots + x_n - 1)).$$

We then can form simplicial sets like $Fr(\Delta^n_k \times X, S)$ with S a pointed simplicial set regarded as a simplicial smooth scheme.

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Theorem (G.–Panin (2010-2018))

$\Omega^\infty \Sigma^\infty S^n_{top} \sim Fr(\Delta^n_{\mathbf{C}}, S^n)$ for any $n > 0$. The topological sphere spectrum is equivalent to

$$M_{fr}(\mathrm{pt})(\mathrm{pt}) := (Fr(\Delta^n_{\mathbf{C}}, S^0), Fr(\Delta^n_{\mathbf{C}}, S^1), Fr(\Delta^n_{\mathbf{C}}, S^2), \dots).$$

Motivic Thom spectra

By definition, a motivic T-spectrum E is called a *Thom spectrum* if every space E_n has the form

$$E_n = \operatorname{colim}_i E_{n,i}, \;\; E_{n,i} = V_{n,i}/(V_{n,i} - Z_{n,i}),$$

where $V_{n,i} \to V_{n,i+1}$ is a directed sequence of smooth varieties, $Z_{n,i} \to Z_{n,i+1}$ is a directed system of smooth closed subschemes in $V_{n,i}$.

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Voevodsky defined the algebraic cobordism spectrum MGL by $MGL(n) := \operatorname{colim}_N Th(E_{n,N})$, where $E_{n,N}$ is the universal bundle over the Grassmannian $G(n, N)$. If $k = \mathbf{C}$ then its realisation $Re(MGL)$ in SH is isomorphic to the complex cobordism spectrum MU.

Definition

For any reasonable symmetric motivic Thom spectrum E one can define *E-framed correspondences* $Fr_*^E(k)$. It has the same objects as Sm/k. Its morphisms are sets $\sqcup_{n\geq 0} Fr_n^E(X,Y)$, where

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As above,

$$Fr^E(X,Y) := \mathrm{colim}(Fr_0^E(X,Y) \xrightarrow{\sigma_Y} Fr_1^E(X,Y) \to \cdots).$$

The *E-framed motive of $Y \in Sm/k$* is defined by

$$M_E(Y) = (Fr^E(\Delta_k^\bullet \times -, Y), Fr^E(\Delta_k^\bullet \times -, Y \otimes S^1), \ldots).$$

For $E \in SH(k)$ (respectively $E \in SH$) and a positive integer N, we let E/N denote an object of $SH(k)$ (respectively $E/N \in SH$) that fits into a triangle $E \xrightarrow{N \cdot id} E \to E/N \to E[1]$.

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Theorem

Let $k = \mathbf{C}$. Suppose E is a symmetric Thom T-spectrum with the bounding constant $d \leqslant 1$ and contractible alternating group action (e.g. $E = MGL$). Then for all integers $N > 1$ the realization functor $Re : SH(\mathbf{C}) \to SH$ induces an isomorphism in SH

$$M_E(\mathrm{pt})(\mathrm{pt})/N \cong Re(E)/N,$$

where $\mathrm{pt} = \operatorname{Spec} \mathbf{C}$ and $M_E(\mathrm{pt})$ is the E-framed motive of pt. In particular, $M_{MGL}(\mathrm{pt})(\mathrm{pt})/N \cong MU/N$ and

$$\Omega^{\infty-1}(MU/N) \sim Fr^{MGL}(\Delta^{\bullet}_{\mathbf{C}}, S^1)/N.$$

As the realization of MGL is isomorphic to MU in SH, the complex cobordism S^2-spectrum, and, by Quillen's Theorem, $\pi_*(MU)$ is isomorphic to the Lazard ring $Laz = \mathbf{Z}[x_1, x_2, \ldots]$, $\deg(x_i) = 2i$, the preceding theorem implies the following

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Corollary

Let $k = \mathbf{C}$. For all $n > 1$ and $i \in \mathbf{Z}$, there is an isomorphism $\pi_i(M_{MGL}(pt)(pt); \mathbf{Z}/n) \cong Laz_i/nLaz_i$, where $M_{MGL}(pt)$ is the MGL-motive of the point $pt = \mathrm{Spec}(\mathbf{C})$.

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Without changing stable homotopy type the MGL-framed motives, and hence the S^2-spectrum $M_{MGL}(pt)(pt)$, can considerably be simplified. It is based on l.c.i. subschemes.

Definition

For $X, Y \in Sm_k$ denote by $Emb_n(X, Y)$ the set of couples (Z, f), where Z is a closed l.c.i. subscheme in $\mathbf{A}^n_X$, finite and flat over X, and f is a regular map $f: Z \to Y$. Note that $Emb_n(X, Y)$ is pointed at the couple $(\emptyset, \emptyset \to Y)$.

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The natural inclusions of affine spaces $\mathbf{A}^n \to \mathbf{A}^{n+1}$ induce stabilization maps of pointed sheaves $Emb_n(-,Y) \to Emb_{n+1}(-,Y)$. Denote by $Emb(-,Y)$ the pointed sheaf $Emb(-,Y) = \operatorname{colim}_n Emb_n(-,Y)$. The forgetful maps $\widetilde{Fr}_n^{MGL}(-,Y) \to Emb_n(-,Y)$ are consistent with the stabilization maps.

Theorem

The MGL-framed motive $M_{MGL}(Y)$ of $Y \in Sm/k$ is locally equivalent to

$$(Emb(\Delta^{\bullet}_k \times -, Y), Emb(\Delta^{\bullet}_k \times -, Y \otimes S^1), \dots).$$

In particular, if $k = \mathbf{C}$ then the S^2-spectrum MU/N is isomorphic in SH to

$$(Emb(\Delta^{\bullet}_{\mathbf{C}}, S^0), Emb(\Delta^{\bullet}_{\mathbf{C}}, S^1), \dots)/N, \quad N > 1.$$

and

$$\Omega^{\infty-1}(MU/N) \sim Emb(\Delta^{\bullet}_{\mathbf{C}}, S^1)/N.$$

We can also compute homology of E-framed motives $M_E(Y)$. Namely the spectrum $H\mathbf{Z} \wedge M_E(Y)(X)$ is computed by the complex $\mathbf{Z}F^E(\Delta_k^\bullet \times X, Y)$ whose chains in each degree are free Abelian groups generated by stable E-framed correspondences from $Fr^E(\Delta_k^n \times X, Y)$ with connected support.

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If $E = MGL$ then we can considerably simplify homology of $M_{MGL}(Y)(X)$. Precisely, it is computed by the complex $\mathbf{Z}Emb(\Delta^\bullet_k \times X, Y)$ whose chains in each degree are free Abelian groups generated by elements of $Emb(\Delta^n_k, \mathrm{pt})$ with stable l.c.i. Z-s.

In particular $H\mathbf{Z} \wedge MU/N$ is computed as the complex $\mathbf{Z}Emb(\Delta^\bullet_{\mathbf{C}}, \mathrm{pt})/N$.