Valleys for the Stochastic Heat Equation

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5JV8

SHE $\quad \partial_t u = \partial_x^2 u + u\,\xi(t,x)$

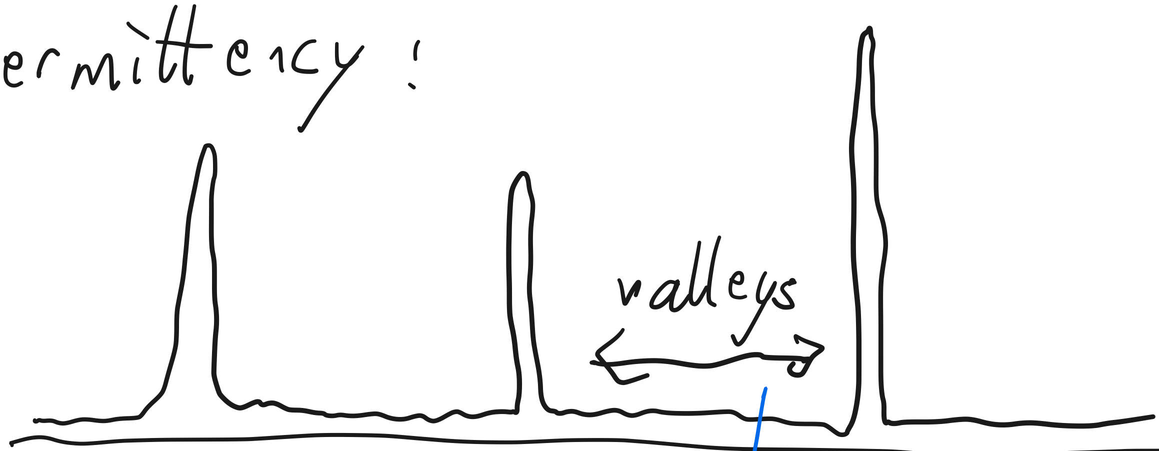
$t \geq 0, \quad x \in \mathbb{R}$

$u(0,x) = u_0(x) = 1$

σSHE $\quad \partial_t u = \partial_x^2 u + \sigma(u)\,\xi(t,x)$

$c, C > 0 \qquad cu \lesssim |\sigma(u)| \lesssim Cu$

Intermittency!



Peaks are well studied

Valleys, not so much

- Ghosal and Yi

  macroscopic Hausdorff dimension of $\{x : u(t,x) \lesssim e^{-ct}\}$

- Corwin and Ghosal studied

  $P(u(t,x) \lesssim s)$ for t, x fixed as $s \downarrow 0$

- Ghosal and Lin, for large t

  $u(t,0) \approx e^{-\frac{t}{24}}$

These results use integrable prob., not available for σSHE. Mostly, only apply to a single point x.

Use mild form of σSHE

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\,\sigma(u(s,y))$$

$\cdot\ \xi(dy\,ds)$

Let
$$L_\sigma = \inf_{a \in \mathbb{R}\setminus\{0\}} \left| \frac{\sigma(a)}{a} \right|$$

$$\mathrm{Lip}_\sigma = \sup_{\substack{a,b \in \mathbb{R} \\ a \neq b}} \left| \frac{\sigma(b) - \sigma(a)}{b-a} \right|$$

Thm 1 $\exists$ constants $\lambda_1, \lambda_2 > 0$ depending on $L_\sigma, \mathrm{Lip}_\sigma$ s.t.

If $R(t) = \exp(\lambda_1 t^{\frac{1}{3}})$

then
$$\sup_{|x| < R(t)} u(t,x) \leq \exp(-\lambda_2 t^{\frac{1}{3}})$$

for large t.

Thm 2 Suppose v satisfies σSHE with $v_0 \geq 0$ satisfying

$$\limsup_{|x| \to \infty} x \log v_0(x) < 0$$

Then, $\exists\, \Lambda_3$ dep. on $L_\sigma, \mathrm{Lip}_\sigma$ and $\exists$ an a.s. finite random time T s.t. if $t > T$ then

$$\sup_{x \in \mathbb{R}} v(t,x) \leq \exp\left(-\Lambda_3 t^{\frac{1}{3}}\right)$$

Some ideas for the proof

1. Decompose the solution into pieces
2. Show that the total mass of (most) pieces tends to 0, study the rate.
3. Use norm estimates (L^1 vs L^∞) to control continuity of solutions and hence the supremum

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① Partitión mass:

SHE $\partial_t u = \partial_x^2 u + u\xi$

is linear

σSHE $\partial_t u = \partial_x^2 u + \sigma(u)\xi$

is not linear

$$\partial_t u = \partial_x^2 u + u \frac{\sigma(u)}{u}\xi$$

$\tilde{\xi}$ (is a worthy mart. meas.)

σSHE is linear, but driven by $\tilde{\xi}$ which depends on u.

Recall $u_0 \equiv 1$

Let $u_0 = \sum_{i=-M}^{M-1} v_0^{(i)} + v_0^{(M)}$

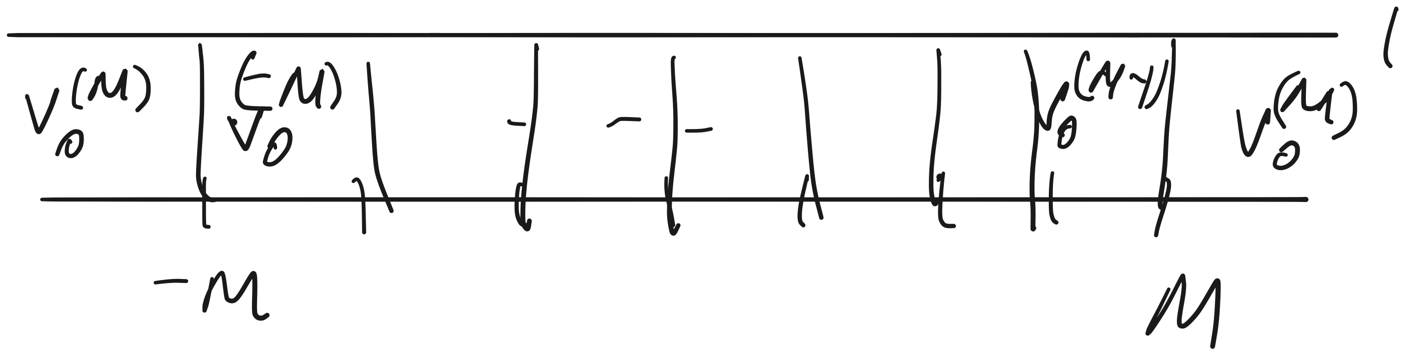
$v_0^{(i)}$ supported on $[i, i+1]$

$V_0^{(M)}$ supported on $[-M,M]^c$

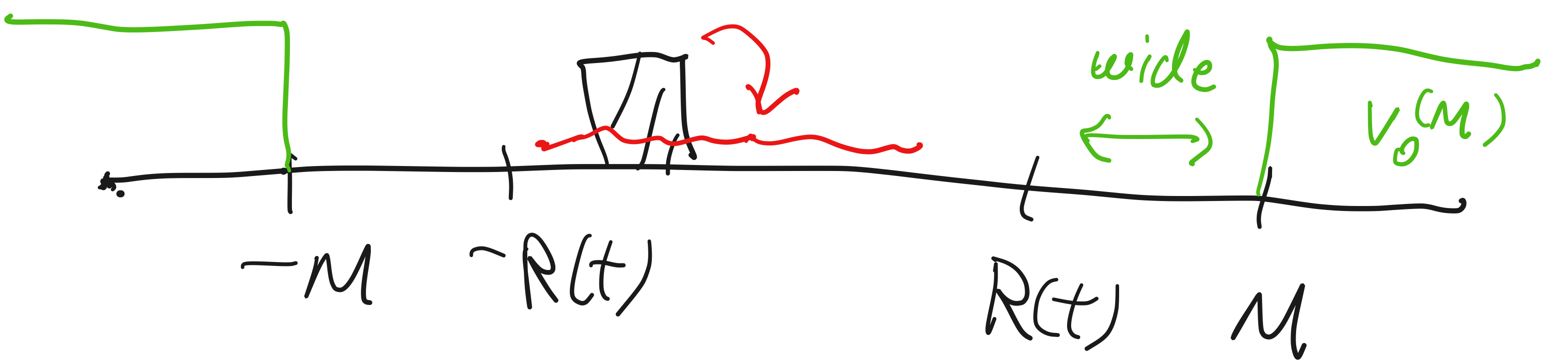
$$\partial_t v^{(i)} = \partial_x^2 v^{(i)} + v^{(i)} \tilde{\xi}(t,x,i)$$

Then

$$u(t,x) = \sum_{i=-M}^{M-1} v^{(i)}(t,x) + v^{(M)}(t,x)$$



Choose $M = M(t) \gg \underbrace{\exp(\lambda_1 t^3)}_{R(t)}$



By time T, not much mass will move from $[-M,M]^c$ to $[-R(t), R(t)]$

Total Mass

Earlier results on total mass.

1. Liggett's book, linear systems (particle systems)

2. Chen, Cranston, Khoshnevisan, Kim adapted this technique to SHE.

Suppose v (such as $v^{(i)}$) satisfies SHE ($v_0 \geq 0$) $\Rightarrow v(t,x) \geq 0$

$$v(t,x) = \int_{\mathbb{R}} G_t(x-y)\, v_0(y)\, dy$$

$$+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\, v(s,y)\, \xi(dy\,ds))$$

$$\text{total mass} = \| v(t) \|_{L^1(dx)} = \| v_0 \|_{L^1}$$

$$+ \int_0^t \int_{\mathbb{R}} v(s,y)\, \xi(dy\,ds)$$

Let $M_t = \| v(t) \|_{L^1}$ (is a mart.)

$$\langle M \rangle_t = \int_0^t \int_{\mathbb{R}} v^2(s,y)\, dy\,ds$$

$$= \int_0^t \| v(s) \|_{L^2}^2\, ds$$

By Ito's lemma

$$d M^{\frac{1}{2}} = \frac{1}{2} M^{-\frac{1}{2}} dM - \frac{1}{4} M^{-\frac{3}{2}} d\langle M \rangle$$

$$= \frac{1}{2} M^{-\frac{1}{2}} dM - \frac{1}{4} M^{\frac{1}{2}} \frac{\| v \|_{L^2}^2}{\| v \|_{L^1}^2} \cdot dt$$

Further progress requires analysis of this term. $H(t)$

Let $f(t) = E\left[M_t^{\frac{1}{2}}\right]$

Get

$$\frac{d}{dt} f(t) \leq -\frac{1}{4} E\left[M_t^{\frac{1}{2}} H(t)\right]$$

We finally get

$$E\left[\|v(t)\|_{L^1}^{\frac{1}{2}}\right] = f(t) \leq C_1 \exp\left(-\frac{t^{\frac{1}{3}}}{C_1}\right)$$