Automorphic representations and L-functions

Introductory talk, part I

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Oaxaca, November 2022

What is an L-function?

At first approximation, an L-function is a Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_n \in \mathbb{C}, \quad a_1 = 1$$

that has three properties

- $D(s)$ converges absolutely in some right-half plane $\Re s \gg 0$. Equivalently, a_n has polynomial growth.
- $D(s)$ admits factorization as an Euler product

  $$D(s) = \prod_p D_p(s), \quad D_p(s) = \sum_{m=0}^{\infty} \frac{a_{p^m}}{p^{ms}}.$$

  In other words, the sequence a_n is multiplicative:

  $$a_{nm} = a_n a_m \text{ whenever } n \text{ and } m \text{ are coprime.}$$

- $D(s)$ has meromorphic continuation to $\mathbb{C}$ with finitely many poles.

Examples

- Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

Euler proved this relation (1737) and used it show that $\sum_p \frac{1}{p} = \infty$. He also evaluated $\zeta(2k)$ in terms of Bernoulli numbers.

Riemann (1859) considered $\zeta(s)$ as a function of a complex variable, proved meromorphic continuation and functional equation. He made the connection to the distribution of primes numbers, paving the way to the eventual proof of the Prime Number Theorem by Hadamard and de la Vallée Poussin (1896, independently). Finally, he also formulated the Riemann Hypothesis.

Examples (cont.)

- Dirichlet L-functions (1837)

$$L(s,\chi) = \sum_{n\in\mathbb{N}|(n,N)=1} \frac{\chi(n)}{n^s} = \prod_{p\nmid N}(1-\chi(p)p^{-s})^{-1}$$

where χ is a Dirichlet character, i.e., a group homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

for some integer N, viewed as a periodic function on $\mathbb{Z}$.
Dirichlet proved (in a quantitative form) the existence of infinitely many primes in any arithmetic progression $a+nd$, $n\in\mathbb{N}$ with $(a,d)=1$.
A key step in the proof was to relate the statement to the non-vanishing of $L(1,\chi)$ for every Dirichlet character.
This was the first time representation theory (of finite abelian groups) was used.

Examples (cont.)

- Dedekind ζ-function (1877)
  F is a number field

$$\zeta_F(s) = \sum_I N(I)^{-s} = \prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1}$$

where I (resp., $\mathcal{P}$) ranges over the ideals (resp., prime ideals) of $\mathcal{O}_F$.

If F is a quadratic field of discriminant D, then

$$\zeta_F(s) = \zeta(s) L(s, \chi_D)$$

where χ_D is the quadratic Dirichlet character $\left(\frac{D}{\cdot}\right)$ (Kronecker symbol).

Examples (cont.)

- Hecke L-function of Hecke characters (1918–20).

$$L(s,\chi) = \sum_{(I,\mathfrak{m})=1} \frac{\chi(I)}{N(I)^s} = \prod_{\mathcal{P}\nmid\mathfrak{m}} (1-\chi(\mathcal{P})N(\mathcal{P})^{-s})^{-1}.$$

An example of a Hecke character with respect to the field $F = \mathbb{Q}[\mathrm{i}]$ is

$$\chi((z)) = \left(\frac{z}{|z|}\right)^{4k}, \quad z = a + b\mathrm{i} \in \mathbb{Z}[\mathrm{i}], \; k \in \mathbb{Z}.$$

Hecke used these L-functions to prove that the angles φ of Gaussian primes are equidistributed in $[0, 2\pi]$.

$$a + b\mathrm{i} = \sqrt{p}e^{\mathrm{i}\varphi}, p = a^2 + b^2, \quad p \equiv 1 \pmod 4.$$

Examples (cont.)

- Artin's L-function (1923–1931). F is a number field,

$$\rho : \mathrm{Gal}(F/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$$

an n-dimensional representation of the Galois group of F.

$$L(s,\rho) = \prod_p \det(1 - p^{-s}\rho(\mathrm{Fr}_p))^{-1}.$$

Artin's L-function corresponding to one-dimensional characters coincide with Dirichlet L-functions. (Kronecker (1853), Hilbert (1896))
The more general statement for one-dimension characters of $\mathrm{Gal}(E/F)$ is Class Field Theory (of which Artin is one of the main architects).

In general, Artin showed, by a group-theoretic argument, that a suitable power of $L(s,\rho)$ is a product of Hecke L-functions and their inverses. Later on, Brauer (1947) refined it to show the same holds for $L(s,\rho)$ itself. In particular, this gives meromorphic continuation and functional equation. However, to date, finiteness of poles (Artin's conjecture) is wide open.

Theorem (Chebotarev's density theorem, 1922)

Let F be a finite Galois extensions of $\mathbb{Q}$ and let $G = \mathrm{Gal}(F/\mathbb{Q})$. Then, for every conjugacy class C in G,

$$\#\{p \le X \mid \mathrm{Fr}_p = C\} \sim \frac{\#C}{\#G} \cdot \frac{X}{\log X} \text{ as } X \to \infty.$$

For cyclotomic number fields, this recovers Dirichlet's theorem.
The following weaker statement is due to Frobenius (1896):

Suppose that F is the splitting field of a separable polynomial $f(x) \in \mathbb{Z}[x]$ of degree n.
For every p, the splitting type of f at p is the partition of n formed by the degrees of the irreducible polynomials in the factorization of f modulo p.
For every $g \in G$ let $c(g)$ be the cycle type of g as a permutation of the roots of f. Again, it is a partition of n. Then, for any partition Π of n,

$$\#\{p \le X \mid \Pi \text{ is the splitting type of } f \text{ at } p\} \sim \frac{\#\{g \in G \mid c(g) = \Pi\}}{\#G} \frac{X}{\log X}$$

Examples (cont.)

- Hecke L-functions (1936)

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

of a modular form f with respect to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, which is an eigenfunction under the Hecke operators. The a_n's are the Fourier coefficients of f in the q-expansion.

Hecke and Artin overlapped in Hamburg during the period 1922–1937. However, they never made the connection between their L-functions..

Examples (cont.)

- Tate's thesis (1950) under Artin reproved the analytic continuation of L-functions of Hecke characters in the adelic language, in which a Hecke character simply becomes a (continuous) character of $F^*\backslash\mathbb{I}_F$.

The key ingredient is the zeta integral

$$\zeta(\Phi,\chi,s)=\int_{\mathbb{I}_F}\Phi(x)\chi(x)\left|x\right|^s\ dx,\quad \Phi\in\mathcal{S}(\mathbb{A}_F).$$

It coincides with $L(\chi,s)$ for suitable data. Following Riemann (and Hecke)

$$\zeta(\Phi,\chi,s)=\int_{|x|\geq 1}\sum_{\xi\in F^\times}\Phi(x\xi)\chi(x)\left|\det(x)\right|^s\ dx-\delta_\chi\frac{\Phi(0)}{s}$$

$$+\int_{|x|\geq 1}\sum_{\xi\in F^\times}\hat{\Phi}(x\xi)\chi(x)^{-1}\left|\det(x)\right|^{1-s}\ dx+\delta_\chi\frac{\hat{\Phi}(0)}{s-1}=\zeta(\hat{\Phi},\chi^{-1},1-s)$$

where $\hat{\Phi}(x)=\int_{\mathbb{A}_F}\Phi(y)\psi(xy)\ dy$ and $\delta_\chi=1$ if $\chi\equiv 1$ and 0 otherwise.

Local theory: local zeta integrals and functional equations.

Examples (cont.)

- The standard L-function of a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$.

$$L(s,\pi) = \prod_{p \nmid N} L(s,\pi_p)$$

where $\pi \simeq \otimes \pi_p$ and for $p \nmid N$

$$\pi_p = \mathrm{Ind}_{B_n(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \left(\begin{pmatrix} t_1 & * & * & * \\ & t_2 & * & * \\ & & \ddots & * \\ & & & t_n \end{pmatrix} \mapsto |t_1|^{s_{1,p}} \dots |t_n|^{s_{n,p}} \right)$$

$$L(s,\pi_p) = \prod_{i=1}^{n} (1 - p^{-(s+s_{i,p})})^{-1}$$

Analytic continuation and functional equation Godement–Jacquet (1972).

Examples (cont.)

- Once again, $\pi = \otimes_p \pi_p$ is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$.

$$\pi_p = \mathrm{Ind}_{B_n(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \left(\begin{pmatrix} t_1 & * & * & * \\ & t_2 & * & * \\ & & \ddots & * \\ & & & t_n \end{pmatrix} \mapsto |t_1|^{s_{1,p}} \ldots |t_n|^{s_{n,p}} \right)$$

Let $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ be an algebraic representation of GL_n.

$$L(s, \pi, \rho) = \prod_{p \nmid N} \det(1 - p^{-s} \rho(\mathrm{diag}(p^{-s_{1,p}}, \ldots, p^{-s_{n,p}})))^{-1}$$

Meromorphic continuation, functional equations and finiteness of poles are parts of (the completely wide open) Langlands conjectures.

For $n = 2$, $a_p = \alpha_p + \alpha_p^{-1}$ (normalized Hecke eigenvalues)

$$L(s, \pi, \mathrm{Sym}^n) = \prod_p \prod_{i=0}^{n} (1 - p^{-s} \alpha_p^{2i-n})^{-1}$$

For modular forms analytic continuation is known Newton–Thorne (2021)

Examples (cont.)

- The Hasse–Weil L-function of an elliptic curve $E/\mathbb{Q}$ (1955).

$$L(E,s) = \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \quad a_p = p + 1 - \#E(\mathbb{F}_p).$$

If E has complex multiplication, then $L(E,s)$ is equal to an L-function of a Hecke character (Shimura, 1971).

If E is semistable, Wiles, Taylor–Wiles (1995) proved analytic properties of $L(E,s)$ by identifying it with a Hecke L-function of a modular form of weight two, solving the Taniyama–Shimura–Weil modularity conjecture (and consequently, Fermat's Last Theorem by earlier ideas and work of Hellegouarch, Frey, Serre, Ribet (1975–90)).

The proof of the T-S-W conjecture in the general case was completed by Breuil–Conrad–Diamond–Taylor (2001).

Examples (cont.)

- More general L-functions for smooth projective varieties $/\mathbb{Q}$, or more generally motives..

Meromorphic continuation only known in special cases, for instance Hasse–Weil L-function of elliptic curves over CM fields (Allen–Calegari–Caraiani–Gee–Helm–LeHung–Newton–Scholze–Taylor–Thorne, 2022), generalizing earlier works by Barnet-Lamb, Clozel, Gee, Geraghty, Harris, Shephard-Barron, Taylor among others.

The proof of this state-of-the-art result involves everything under the sun.

Even in these cases, finiteness of poles is not known.

A non-example

$$\sum_{D=1}^{\infty} \frac{H(D)}{D^s}$$

where $H(D)$ is the Hurwitz class number of positive definite binary quadratic forms of discriminant $-D$.
It admits a meromorphic continuation and a functional equation, but it does not admit an Euler product (nor is it a linear combination of L-functions in the traditional sense). (Shintani, 1975)
An L-function is not the same thing as a ζ function

Remarks

- Nobody knows how to meromorphically continue an Euler product *per se*.
- (Estermann's dichotomy (1926))
  Let $f \in \mathbb{Z}[X]$ with $f(0) = 1$. Consider the Euler product

  $$L(s) = \prod_p f(p^{-s})^{-1}, \quad \Re s > 1.$$

  Then, the following conditions are equivalent.
  1. The roots of f all lie on the unit circle.
  2. f is a product of cyclotomic polynomials.
  3. There exist integers $a_1, \dots, a_k > 0$ and signs $\epsilon_1, \dots, \epsilon_k = \pm 1$ such that

     $$L(s) = \prod_{i=1}^{k} \zeta(a_i s)^{\epsilon_i}.$$

  4. $L(s)$ admits a meromorphic continuation to $\mathbb{C}$.

  Otherwise, $L(s)$ has natural boundary $\Re s = 0$.

Langlands philosophy (crude formulation)

Every L-function is the standard L-functions of an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$.
In particular,

1. $D(s) = \prod_p D_p(s)$ where for every p, $D_p(s) = P_p(q^{-s})^{-1}$ where $P_p \in \mathbb{C}[X]$ is a monic polynomial of degree $\leq n$.
2. For almost all p, D_p is of degree n
3. There exist $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that

$$L(s) = \big(\prod_{i=1}^{n} \Gamma_{\mathbb{R}}(s+\alpha_i)\big) \cdot D(s) \text{ where } \Gamma_{\mathbb{R}}(s) = \int_{\mathbb{R}} e^{-\pi t^2} |t|^s \frac{dt}{t}$$

   satisfies the functional equation

$$L(s) = \epsilon Q^{\frac{1}{2}-s} L^{\vee}(1-s)$$

   where $L^{\vee}$ is a function of a similar type and $Q \in \mathbb{N}$.
4. In the "unitary" case (say, π is cuspidal unitary) $L^{\vee}(s) = \overline{L(\overline{s})}$.

Equidistribution and L-functions, à la Serre (1968)

Definition

Let μ be a probability Radon measure on a compact topological space X. A sequence $x_1, x_2, \ldots$ of elements of X is μ-equidistributed if for every continuous function f on X

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \to \mu(f) \text{ as } n \to \infty.$$

It is enough to check this condition for test functions f which span a dense subspace of $C(X)$.

Basic example: G a compact group; X the set of conjugacy classes of G; μ the pushforward to X of the normalized Haar measure on G. Then,

$$x_n \text{ is equidistributed in } X \iff \frac{1}{n}\sum_{i=1}^{n} \chi(x_i) \to 0$$

for every irreducible character $\chi \neq 1$ of G. (For $G = X = S^1$, Weyl 1916)

Theorem

Let x_p be elements of G (p prime). Suppose that for every irreducible representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, $\rho \neq 1$ the Euler product

$$L(\rho, s) = \prod_p \det(1 - p^{-s}\rho(x_p))^{-1}$$

(which converges for $\Re s > 1$) has analytic continuation to $\Re s \geq 1$ with no zeros. Then, x_p are equidistributed in X.

Corollary

Let π be a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$. Suppose that for all $n \geq 1$, $L^S(s, \pi, \mathrm{Sym}^n)$ is holomorphic and non-zero for $\Re s \geq 1$. Then, writing $a_p = e^{i\theta_p} + e^{-i\theta_p}$, $0 \leq \theta_p \leq \pi$, the θ_p are equidistributed with respect to the measure $\frac{dx}{\pi} - \frac{\cos 2x}{\pi}\, dx$.

Proof.

Write $x_{p^m} = x_p^m$ for $m \geq 1$. If χ is the character of ρ, then

$$-\frac{L'}{L} = \sum_{n=1} \frac{\chi(x_n)\Lambda(n)}{n^s}$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, m \geq 1 \\ 0 & \text{otherwise} \end{cases}$ is von Mangoldt function.

Since $\frac{L'}{L}$ is holomorphic for $\Re s \geq 1$, by a standard Tauberian theorem

$$\frac{1}{N} \sum_{n=1}^{N} \chi(x_n)\Lambda(n) \to 0 \text{ as } N \to \infty.$$

By Abel summation, this is equivalent to

$$\frac{1}{\#\{p : p \leq N\}} \sum_{p \leq N} \chi(x_p) \to 0 \text{ as } N \to \infty.$$

Remarks

The above theorem is the starting point of all previously discussed equidistributions results (from residue classes of primes, through angles of Gaussian primes, Frobenius classes and Hecke eigenvalues).

Of course the hard part is to prove the analytic properties of the relevant L-functions. I will say a little about that in the next part.

Moreover, any sharper results on the rate of convergence can in principle be formulated (and sometimes proved) in terms of zero-free regions (or sparsity results on the number of zeros) of L-functions.

A notable example is the Bombieri–Vinogradov theorem on the average (in the modulus) of the error term in the prime number theorem for arithmetic progressions. This is a key ingredient in Maynard's proof that for every m

$$\liminf_n p_{n+m} - p_n < \infty$$