Automorphic representations and L-functions

Introductory talk, part II

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L-functions and the Ramanujan Hypothesis

Theorem (Langlands 1970)

Let π be a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$*.*

Assume that $L^S(s, \pi, \mathrm{Sym}^n)$ is holomorphic for $\Re s > 1$ for all $n \geq 1$.

Then $|a_p| \leq 2$ for all p.

Recall that a_p is the eigenvalue of the normalized Hecke operator, and writing $a_p = \alpha_p + \alpha_p^{-1}$

$$L(s, \pi, \mathrm{Sym}^n) = \prod_p \prod_{i=0}^n (1 - p^{-s} \alpha_p^{2i-n})^{-1}$$

Proof.

As representations of SL_2 we have

$$\mathrm{Sym}^n \otimes \mathrm{Sym}^n = \oplus_{i=0}^n \mathrm{Sym}^{2i}.$$

It follows that

$$L^S(s,\pi,\mathrm{Sym}^n \otimes \mathrm{Sym}^n) = \prod_{i=0}^n L^S(s,\pi,\mathrm{Sym}^{2i})$$

and hence, $L^S(s,\pi,\mathrm{Sym}^n \otimes \mathrm{Sym}^n)$ is holomorphic for $\Re s > 1$.
On the other hand, $L^S(s,\pi,\mathrm{Sym}^n \otimes \mathrm{Sym}^n)$ has non-negative coefficients.
Therefore, by Landau's Lemma, the Euler product converges absolutely.
In particular, for every p, $\det(1 - p^{-s}(\mathrm{Sym}^n \otimes \mathrm{Sym}^n)(\begin{pmatrix} \alpha_p & \\ & \alpha_p^{-1} \end{pmatrix}))^{-1}$ is holomorphic for $\Re s > 1$, where $a_p = \alpha_p + \alpha_p^{-1}$.
In other words, the eigenvalues of $(\mathrm{Sym}^n \otimes \mathrm{Sym}^n)(\begin{pmatrix} \alpha_p & \\ & \alpha_p^{-1} \end{pmatrix})$ are $\le p$ in absolute value. This means that $|\alpha_p|^{\pm 2n} \le p$.
Since this is true for all n, we conclude that $|\alpha_p| = 1$. □

Eisenstein series

Let $\mathbb{H} = \{x + \mathrm{i}y \mid y > 0\}$ be the hyperbolic upper half plane.
The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ simply transitively by Möbius transformations.
The stabilizer of the point i is $K = SO(2)$.
The function $\mathrm{Im}\, z$ is invariant under $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$.
Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_\infty = N \cap \Gamma$.
The simplest Eisenstein series (Maass, 1949) is

$$E(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^{s+\frac{1}{2}} = \sum_{(m,n) \in \mathbb{Z}^2 \mid \gcd(m,n)=1} \frac{(\mathrm{Im}\, z)^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}$$

The series converges absolutely for $\Re s > 1$ and defined a function on $\Gamma \backslash \mathbb{H}$.
The constant term $\int_{\Gamma_\infty \backslash N} E(nz;s)\, dn$ is given by

$$(\mathrm{Im}\, z)^{s+\frac{1}{2}} + \frac{\zeta^*(2s)}{\zeta^*(2s+1)} (\mathrm{Im}\, z)^{-s+\frac{1}{2}}$$

In the 1960's Langlands computed the constant term of Eisenstein series induced from a cuspidal representation π of $M(\mathbb{A})$ where M is the Levi part of a maximal parabolic subgroup of G defined over $\mathbb{Q}$.

The main term is

$$\prod_{i=1}^{m} \frac{L(is, \pi, r_i)}{L(is+1, \pi, r_i)}$$

where r_i are certain representations of the L-group of M.

This computation led Langlands to the notion of the L-group, the general notion of an automorphic L-function and finally to the functoriality conjectures (roughly: "every L-function is a standard L-function of GL_n").

Examples for r_i

- $G = \mathrm{GL}_{n_1+n_2}$, $M = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, $r_1 = \mathrm{St}_{n_1} \otimes \mathrm{St}_{n_2}$ (Rankin–Selberg convolution).
- G is an orthogonal group of rank n, $M = \mathrm{GL}_n$

$$r_1 = \begin{cases} \mathrm{Sym}^2 & \text{if } G = SO(2n+1) \\ \wedge^2 & \text{if } G = SO(2n) \end{cases}$$

- G – a classical group of rank n, $M = G' \times \mathrm{GL}_{n-k}$ where G' is a classical group (of the same type) of rank k, $r_1 = \mathrm{Can}_k \times \mathrm{St}_{n-k}$ where Can_k is the "canonical" representation of ${}^L G'$.
- $G = G_2$, $M = \mathrm{GL}_2$, $r_1 = \mathrm{Sym}^3$.
- $G = E_8$, $M = \mathrm{GL}_8$, $r_1 = \wedge^3$.
- $G = E_8$, $M \approx \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_5$, $r_1 = \mathrm{St}_2 \times \mathrm{St}_3 \times \mathrm{St}_5$.

The theory of Eisenstein series was developed by Selberg and Langlands (1950's-60's) in connection with the spectral theory of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Thanks to Bernstein, there is now a simple proof of the meromorphic continuation of Eisenstein series.

This gives the meromorphic continuation of $L(s, \pi, r_i)$, $i = 1, \dots, m$.

Unfortunately, by itself it gives neither the functional equation nor information on the poles.

In the important case where G is quasi-split and π is generic, Shahidi's work (1978–2000) gives a functional equation and *some* information about the poles.
This eventually leads to Sym^4 functoriality from GL_2 to GL_5 as well as functoriality from generic representations of classical groups to GL_N. (Cogdell–Kim–Piatetskii-Shapiro–Shahidi, 2002)

Basic Eisenstein series

Consider the right action of GL_n on row vectors with n entries.
For any $\Phi \in \mathcal{S}(\mathbb{A}^n)$ the Eisenstein series

$$\mathcal{E}(\Phi, g, s) = \int_{\mathbb{Q}^* \backslash \mathbb{I}} \sum_{\xi \in \mathbb{Q}^n \setminus \{0\}} \Phi(t\xi g) \left|\det(tg)\right|^{s+\frac{1}{2}} \, dt, \quad g \in [\mathrm{GL}_n]$$

converges for $\Re s > \frac{1}{2}$ and can be meromorphically continued to $\mathbb{C}$.

Riemann (followed up by Hecke and Tate)

$$\mathcal{E}(\Phi, g, s) = \int_{|t| \geq 1} \sum_{\xi \in \mathbb{Q}^n \setminus \{0\}} \Phi(t\xi g) \left|\det(tg)\right|^{s+\frac{1}{2}} \, dt - \frac{\Phi(0)}{n(s+\frac{1}{2})}$$

$$+ \int_{|t| \geq 1} \sum_{\xi \in \mathbb{Q}^n \setminus \{0\}} \hat{\Phi}(t\xi g^*) \left|\det(tg^*)\right|^{\frac{1}{2}-s} \, dt + \frac{\hat{\Phi}(0)}{n(s-\frac{1}{2})} = \mathcal{E}(\hat{\Phi}, g^*, -s)$$

$$\text{where } \hat{\Phi}(x) = \int_{\mathbb{A}^n} \Phi(y) \psi([x, y]) \, dy, \quad [xg, yg^*] = [x, y].$$

Rankin–Selberg and Jacquet–Shalika integrals

Let $\varphi_i \in V_{\pi_i}$ be cusp forms of $\mathrm{GL}_n(\mathbb{A})$. Then, (Rankin, 1939; Selberg, 1940; Jacquet–Piatetski-Shapiro–Shalika 1980s)

$$\int_{[\mathrm{GL}_n]} \varphi_1(g)\varphi_2(g)\mathcal{E}_\Phi(g,s)\ dg \text{ ``$=$''} L(s+\tfrac{1}{2},\pi_1\times\pi_2)$$

(i.e., up to local integrals).

Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_{2n}(\mathbb{A})$. Then, (Jacquet–Shalika, 1990)

$$\int_{[\mathrm{GL}_n]}\int_{[\mathrm{Mat}_n]} \varphi(\left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix}\right)\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right))\psi(\mathrm{tr}\, X)\ dX\ \mathcal{E}_\Phi(g,s)\ dg \text{ ``$=$''} L(s+\tfrac{1}{2},\pi,\wedge^2)$$

Many other integrals involving Eisenstein series have been discovered over the years. (The uncontested champion here is David Ginzburg.)
The conceptual framework is still far from understood.

Godement–Jacquet integral

Now consider the space Mat_n of $n \times n$-matrices.
Left and right matrix multiplication gives rise to the tensor product representation

$$\iota : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{n^2}$$

The pullback of $\mathcal{E}(\Phi, s)$, $\Phi \in \mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$ to $[\mathrm{GL}_n \times \mathrm{GL}_n]$ is a reproducing kernel for the standard L-function in the sense that

$$(\mathcal{E}(\Phi, \iota(g, h), s))^{\mathrm{cusp}} = \sum_{\pi} \sum_{\varphi} L(ns + \tfrac{1}{2}, \pi) \delta(f_{\Phi,s}) \varphi(g) \overline{\varphi(h)}$$

for a suitable test function $f_{\Phi,s}$ where π ranges over the irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$ and φ over an orthonormal basis in the space of π.
In other words,

$$\langle \mathcal{E}(\Phi, \iota(g, \cdot), s), \varphi \rangle_{[Z \backslash \mathrm{GL}_n]} = L(ns + \tfrac{1}{2}, \pi) \delta(f_{\Phi,s}) \varphi(g)$$

Relation between Godement–Jacquet and Hecke zeta integrals for GL_2 (Ginzburg–Soudry 2020)

Identify the space Mat_2 of 2×2-matrices over $\mathbb{Q}$ with $\mathbb{Q}^4$ via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \leftrightarrow (a, b, c, d).$$

This gives an action of $\mathrm{GL}_4(\mathbb{A})$ on $\mathcal{S}(\mathrm{Mat}_2(\mathbb{A}))$ by right translation. In these coordinates, the tensor representation

$$\iota = \iota_L \times \iota_R : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$$

$$g_1^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)g_2 \leftrightarrow (a, b, c, d)\iota_L(g_1)\iota_R(g_2)$$

is given explicitly as

$$\iota_R(g) = \mathrm{diag}(g, g),\ \ \iota_L(\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right))^{-1} = \begin{pmatrix} \alpha & & \gamma & \\ & \alpha & & \gamma \\ \beta & & \delta & \\ & \beta & & \delta \end{pmatrix}.$$

For $\Phi \in \mathcal{S}(\mathbb{A}^4)$ let $\tilde{\Phi} \in \mathcal{S}(\mathrm{Mat}_2(\mathbb{A}))$ be the partial Fourier transform

$$\tilde{\Phi}(\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)) = \int_{\mathbb{A}^2} \Phi(w, v)\psi([w, u])\, dw, \quad u, v \in \mathbb{A}^2$$

(in the top row) where $[(a, b), (c, d)] = ad - bc$.
By the Poisson summation formula, for any $\Phi \in \mathcal{S}(\mathbb{A}^4)$

$$\sum_{\xi \in \mathrm{Mat}_2(\mathbb{Q})} \omega_\psi(g)\tilde{\Phi}(\xi) = |\det g|^{\frac{1}{2}} \sum_{\xi \in \mathbb{Q}^4} \Phi(\xi g), \quad g \in \mathrm{GL}_4(\mathbb{A})$$

where the representation ω_ψ of $\mathrm{GL}_4(\mathbb{A})$ on $\mathcal{S}(\mathrm{Mat}_2(\mathbb{A}))$ is

$$\omega_\psi(g)\tilde{\Phi}(v) = |\det g|^{\frac{1}{2}} \widetilde{\Phi(\cdot g)}(v), \quad g \in \mathrm{GL}_4(\mathbb{A}),\ v \in \mathrm{Mat}_2(\mathbb{A}).$$

For any $\Phi \in \mathcal{S}(\mathrm{Mat}_2(\mathbb{A}))$ we have

1. $\omega_\psi(\iota_R(g))\Phi(\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)) = \Phi(\left(\begin{smallmatrix} ug^* \\ vg \end{smallmatrix}\right)),\ u, v \in \mathbb{A}^2,\ g \in \mathrm{GL}_2(\mathbb{A})$
2. $\omega_\psi(\iota_L(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)))\Phi(Y) = \psi(x \det Y)\Phi(Y), \quad Y \in \mathrm{Mat}_2(\mathbb{A}),\ x \in \mathbb{A}$

Consider

$$\theta^*_\Phi(g) = \sum_{0 \neq \xi \in \mathrm{Mat}_2(\mathbb{Q})} \omega_\psi(g)\Phi(\xi), \quad g \in \mathrm{GL}_4(\mathbb{A}).$$

For any $\xi \in \mathrm{Mat}_2(\mathbb{Q})$ we have

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \omega_\psi(\iota_L(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)))\Phi(\xi)\psi(x)\ dx = \begin{cases} \Phi(\xi) & \text{if } \det\xi = 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\mathrm{GL}_2(\mathbb{Q})$ acts transitively on

$$\{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \mid [u, v] = 1\} = \{\xi \in \mathrm{Mat}_2(\mathbb{Q}) \mid \det\xi = 1\}$$

by $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} ug^* \\ vg \end{smallmatrix}\right)$. It follows that

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \theta^*_\Phi(\iota_L(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))g)\psi(x)\ dx = \sum_{\gamma \in T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \omega_\psi(\iota_R(\gamma)g)\Phi(I)$$

where $T = \{\left(\begin{smallmatrix} * & \\ & 1 \end{smallmatrix}\right)\}$ is the stabilizer of $\left(\begin{smallmatrix} (1,0) \\ (0,1) \end{smallmatrix}\right)$.

Taking Mellin transform,

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \mathcal{E}(\Phi, \iota_L(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))g, s)\psi(x)\ dx$$

$$= \int_{\mathbb{Q}\backslash\mathbb{A}} \int_{\mathbb{Q}^*\backslash\mathbb{I}} \theta^*_{\tilde{\Phi}}(\iota_R(tI)\iota_L(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))g) \left|\det tg\right|^s\ dt\ \psi(x)\ dx$$

$$= \sum_{\gamma\in T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \int_{\mathbb{Q}^*\backslash\mathbb{I}} \omega_\psi(\iota_R(t\gamma)g)\Phi(I) \left|\det tg\right|^s\ dt.$$

Thus, for any cusp form ϕ on $[Z\backslash \mathrm{GL}_2]$

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \left\langle \mathcal{E}(\Phi, \iota(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \cdot), s), \phi \right\rangle_{[Z\backslash \mathrm{GL}_2]}\ \psi(x)\ dx$$

$$= \int_{T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} \omega_\psi(\iota_R(g))\Phi(I) \left|\det g\right|^{2s} \phi(g)\ dg$$

$$= \int_{T(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \omega_\psi(\iota_R(g))\Phi(I) \int_{\mathbb{Q}^*\backslash\mathbb{I}} \phi(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)g) \left|t\right|^{2s}\ dt\ \left|\det g\right|^{2s}\ dg.$$

The function $\omega_\psi(\iota_R(g))\Phi(I)$ is a test function on $T(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})$.

Recall

$$\langle \mathcal{E}(\Phi, \iota(g,\cdot), s), \phi \rangle_{[Z\backslash \mathrm{GL}_2]} \text{ ``}{=}\text{'' } L(2s+\tfrac{1}{2}, \pi)\phi(g).$$

Taking Whittaker coefficients of both sides we get

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \left\langle \mathcal{E}(\Phi, \iota(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \cdot), s), \phi \right\rangle_{[Z\backslash \mathrm{GL}_2]} \psi(x)\, dx$$

$$\text{``}{=}\text{'' } L(2s+\tfrac{1}{2}, \pi) \int_{\mathbb{Q}\backslash\mathbb{A}} \phi(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))\psi(x)\, dx.$$

On the other hand,

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \left\langle \mathcal{E}(\Phi, \iota(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), \cdot), s), \phi \right\rangle_{[Z\backslash \mathrm{GL}_2]} \psi(x)\, dx$$

$$= \int_{T(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \omega_\psi(\iota_R(g))\Phi(I) \int_{\mathbb{Q}^*\backslash\mathbb{I}} \phi(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) g)\, |t|^{2s}\, dt\, \left|\det g\right|^{2s}\, dg.$$

Thus, the Hecke integral is obtained from the Godement–Jacquet integral by taking a Whittaker coefficient.

Similar analysis shows that for any cuspidal representation π of $G(\mathbb{A})$, $G = \mathrm{GL}_{2n}$, $n \geq 1$ and $\phi \in V_\pi$ we have

$$\int_{[\mathrm{GL}_n]} \phi\left(\begin{pmatrix} g & \\ & h \end{pmatrix}\right) |\det g|^s \ dg$$

$$\text{``}=\text{''} \ L(s+\tfrac{1}{2},\pi) \int_{[\mathrm{Mat}_n]} \phi\left(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} h & \\ & h \end{pmatrix}\right)\psi(\mathrm{tr}\, X) \ dX, \quad h \in [\mathrm{GL}_n].$$

This can also be proved using unfolding (Bump–Furusawa–Ginzburg). Note that the Shalika functional on the right-hand side is not unique. Integrating against an Eisenstein series (in the h variable) we get a relation

$$\int_{[\mathrm{GL}_n \times \mathrm{GL}_n]} \phi\left(\begin{pmatrix} g & \\ & h \end{pmatrix}\right)\mathcal{E}(h,s') \, |\det g|^{s-\frac{1}{2}} \ dg \ dh$$

$$\text{``}=\text{''} \ L(s+\tfrac{1}{2},\pi) \int_{[\mathrm{GL}_n]} \int_{[\mathrm{Mat}_n]} \phi\left(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} h & \\ & h \end{pmatrix}\right)\psi(\mathrm{tr}\, X) \ dX \ \mathcal{E}(h,s') \ dh$$

$$\text{``}=\text{''} \ L(s+\tfrac{1}{2},\pi)L(s'+\tfrac{1}{2},\pi,\wedge^2) \int_{[N]} \phi(u)\psi_N(u) \ du$$

(Jacquet–Shalika, Bump–Friedberg, Friedberg–Jacquet)

More reproducing kernels for L-functions – the doubling method of Piatetski-Shapiro–Rallis

Let V be a finite-dimensional vector space over $\mathbb{Q}$ and h a non-degenerate symmetric or alternating bilinear form on V.
Let $G = \mathrm{Isom}(V, h)$, an orthogonal or symplectic group.
Consider the "doubled" group $H = \mathrm{Isom}(V \oplus V, h \oplus (-h))$.
The diagonal

$$V^{\triangle} = \{(v, v) \mid v \in V\}$$

is a maximal isotropic subspace of $V \oplus V$ defined over $\mathbb{Q}$. Its stabilizer P is a maximal parabolic subgroup of H whose Levi part is isomorphic to $\mathrm{GL}(V)$.
In particular, H is split (even if G isn't).
We can form an Eisenstein series $\mathcal{E}(f, s)$ induced from the character $|\det_V|^s$ of $P(\mathbb{A})$.
We have an embedding

$$\iota : G \times G \hookrightarrow H$$

Doubling method (Piatetski-Shapiro–Rallis, 1987)

The restriction of $\mathcal{E}(f,s)$ to $[G \times G]$ is a reproducing kernel for the L-function pertaining to the canonical representation

$$\mathrm{Can} : {}^L G \to \mathrm{GL}(N),$$

where

$$N = \dim V + \begin{cases} 1 & \text{if } h \text{ is alternating} \\ 0 & \text{if } h \text{ is symmetric and } \dim V \text{ is even} \\ -1 & \text{if } h \text{ is symmetric and } \dim V \text{ is odd} \end{cases}$$

In other words, for any cuspidal $\phi \in V_\pi$

$$\langle \mathcal{E}^*(f, \iota(\cdot, g), s), \phi \rangle_{[G]} \text{ “=” } L(s + \tfrac{1}{2}, \pi, \mathrm{Can})\phi(g).$$

Remarks

- Unlike the case of the general linear group, the meromorphic continuation of $\mathcal{E}(f, s)$ is not merely a consequence of Poisson summation formula, but rather of the general theory of Eisenstein series.
- A related issue is the normalization $\mathcal{E}^*(f, s)$ of $\mathcal{E}(f, s)$. Currently, there is no satisfactory geometric way to do it.
- The doubling method gives meromorphic continuation and functional equation of $L(s, \pi, \mathrm{Can})$. Moreover, the poles are controlled by those of $\mathcal{E}^*(f, s)$ (which have to be studied separately but they do not depend on π – work by Kudla and Rallis in the 1990s).
- Just as with the Godement–Jacquet integral, many other integral representations of $L(s, \pi, \mathrm{Can})$ can be derived by taking appropriate models (Ginzburg, Soudry, Adrianov, Piatetski-Shapiro-Rallis). This was only realized in retrospect (rather recently) by Ginzburg–Soudry.

Generalized doubling (Cai–Friedberg–Ginzburg–Kaplan, 2017)

Let W be another finite-dimensional vector space, $n = \dim W$.
Consider the symmetric bilinear form β on $W \oplus W^\vee$ such that

$$\beta(w, w^\vee) = \langle w^\vee, w \rangle, \quad \beta\big|_{W \times W} \equiv 0, \ \beta\big|_{W^\vee \times W^\vee} \equiv 0.$$

Let

$$H = \mathrm{Isom}(V \otimes (W \oplus W^\vee), h \otimes \beta).$$

For $n = 1$, this is isomorphic to $(V \oplus V, h \oplus (-h))$ considered before.
Note that $V \otimes W$ is a maximal isotropic subspace. Let P be the maximal parabolic subgroup stabilizing $V \otimes W$. Its Levi subgroup M is isomorphic to $\mathrm{GL}(V \otimes W) \simeq \mathrm{GL}(n \cdot \dim V)$.
Fix a cuspidal representation τ of $\mathrm{GL}(W)(\mathbb{A}) \simeq \mathrm{GL}_n(\mathbb{A})$.
Let σ be the corresponding Speh representation of $M(\mathbb{A})$.
Let $\mathcal{E}(\cdot, s)$ be the (suitably normalized) Eisenstein series induced from σ.

Fix a complete flag

$$0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n = W$$

Let $Q = LU$ be the parabolic subgroup of H stabilizing the flag

$$0 \subsetneq V \otimes W_1 \subsetneq \cdots \subsetneq V \otimes W_{n-1}.$$

Thus, $L \simeq \mathrm{GL}(V) \times \cdots \times \mathrm{GL}(V) \times \mathrm{Isom}(V \oplus V, h \oplus (-h))$.
The stabilizer of a suitable generic character ψ_U of U is the image of

$$\iota : G \times G \hookrightarrow L, \quad (g, h) \mapsto \{(g, \ldots, g, (g, h)).$$

Generalized doubling (Cai–Friedberg–Ginzburg–Kaplan, 2017)

As a function on $[G \times G]$, the Fourier coefficient

$$\mathcal{E}^{\psi_U}(f, \cdot, s) = \int_{[U]} \mathcal{E}(f, u\cdot, s)\psi_U(u)\, du$$

is a reproducing kernel for the L-function $L(s, \cdot \otimes \tau, \mathrm{Can} \otimes \mathrm{St})$ i.e. $\forall \phi \in V_\pi$

$$\left\langle \mathcal{E}^{\psi_U}(\iota(\cdot, g), s), \phi \right\rangle_{[G]} \text{``$=$''}\, L(s + \tfrac{1}{2}, \pi \otimes \tau, \mathrm{Can} \otimes \mathrm{St})\phi(g).$$

Remarks

- A similar construction gives a reproducing kernel for the L-function

$$L(s, \pi \otimes \tau) L(s, \pi^\vee \otimes \tau)$$

where π ranges over the cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$.
However, I'm not aware of an explicit reproducing kernel for

$$L(s, \pi \otimes \tau)$$

itself.

- (Double descent, Ginzburg–Soudry, 2022) Taking $n = N$, the residue

$$\mathrm{Res}_{s=\frac{1}{2}} \mathcal{E}^{\psi_U}(f, \cdot, s)$$

is a reproducing kernel for the space of cusps forms that functorially lift to τ, i.e.

$$\mathrm{Res}_{s=\frac{1}{2}} \mathcal{E}^{\psi_U}(f, \iota(g,h), s) \text{``}=\text{''} \sum_{\phi} \phi(g)\overline{\phi(h)}.$$