Simple cuspidals and the Langlands correspondence

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Plan of the talk

1) Introduction
2) Simple cuspidals for $\mathrm{GL}(n, F)$
3) Simple cuspidals for $\mathrm{Sp}(2n, F)$
4) The Langlands correspondence for simple cuspidals

1) Introduction

Let F be a non-Archimedean locally compact field, and G a split reductive group over $\mathbb{Z}$.

I shall focus on cuspidal (irreducible, complex) representations of $G(F)$. They are the building blocks in the theory of smooth representations of $G(F)$.

From Stevens's lectures, you know that a general way to construct them is via induction from an open compact mod. centre subgroup of G.

Actually for any G where all the cuspidals have been constructed, they are so obtained, and in a precise way. That is the case for $\mathrm{GL}(n)$ (Bushnell & Kutzko), classical groups when the residue characteristic p of F is odd (Stevens et al.), and general G provided p is large enough (Yu, Fintzen).

1) Introduction

In 2010, Gross & Reeder invented the simple cuspidals. They exist for any (split) G, and are given by an easy construction which is completely uniform across G and p. For $\mathrm{GL}(n)$ they are special cases of a construction due to Carayol in the 1970's.

I shall first describe them for $\mathrm{GL}(n)$ and Sp(2n), then tell what they give through the local Langlands correspondence, which attaches to a cuspidal for $G(F)$ a morphism of the Weil group W_F of F into the dual group $\widehat{G}$ of G, which is $\mathrm{GL}(n, \mathbb{C})$ when $G = \mathrm{GL}(n)$ and $\mathrm{SO}(2n+1, \mathbb{C})$ when $G = \mathrm{Sp}(2n)$.

2) Simple cuspidals for $GL(n, F)$

General notation :

- $\mathcal{O}_F$ is the ring of integers of F.
- $\mathfrak{p}_F$ its maximal ideal.
- $\kappa = \mathcal{O}_F/\mathfrak{p}_F$, $q = \mathrm{card}(\kappa) = p^f$.
- ϖ a uniformizer of F, $\mathfrak{p}_F = \varpi\mathcal{O}_F$.
- ψ a non-trivial character of κ.

Notation for the general linear group :

$G = \mathrm{GL}(n, F)$ $(n > 1)$: linear automorphisms of F^n, with canonical basis $e_1, \ldots, e_n$. Identify F^* with the centre of G.

- $K = GL(n, \mathcal{O}_F)$.
- I Iwahori subgroup : matrices in K with upper triangular reduction mod. $\mathfrak{p}_F$.
- $N_G(I) = \langle \Pi \rangle\, I$, where $\Pi(e_i) = e_{i+1}$ for $i = 1, \ldots, n-1$, and $\Pi(e_n) = \varpi e_1$. Note that $\Pi^n = \varpi \cdot \mathrm{id}$.

2) Simple cuspidals for $GL(n, F)$

- I^1 the pro-p Iwahori : matrices in I with unipotent reduction mod. $\mathfrak{p}_F$ (i.e. diagonal entries in $1 + \mathfrak{p}_F$).
- I^2 consists of matrices x in I^1 with $x_{i,i+1} \in \mathfrak{p}_F$ for $i = 1, \ldots, n-1$ and $x_{n,1} \in \mathfrak{p}_F^2$.
- $I^1/I^2 = \kappa^n$: send $x \in I^1$ to

$$(x_{1,2} \operatorname{mod.} \mathfrak{p}_F, \ldots, x_{n-1,n} \operatorname{mod.} \mathfrak{p}_F, x_{n,1}/\varpi \operatorname{mod.} \mathfrak{p}_F).$$

- ψ defines a character $(\psi, \ldots, \psi)$ of κ^n, hence a character λ_ψ of I^1.

Theorem

The intertwining set of λ_ψ in G is $J = \langle \Pi \rangle F^* I^1$, which is also its normalizer.

2) Simple cuspidals for $GL(n, F)$

Corollary

If λ is any character of J extending λ_ψ, then $\mathrm{ind}_J^G \lambda$ is a cuspidal representation of G.

Remarks.

1. *Given ψ, λ is determined by its value on Π, and its restriction to U_F, which is trivial on $1 + \mathfrak{p}_F$, hence amounts to a character of κ^*.*
2. *Varying ψ and λ, we get the* **simple** *cuspidals of G.*
3. *We may choose different non-trivial characters $\psi_1, \ldots, \psi_n$ on each coordinate of κ^n, and get cuspidals in the same fashion, but under the action of I/I^1 they are equivalent to the preceding ones. Similarly if we change ϖ.*

3) Simple cuspidals for $\mathrm{Sp}(2n)$

$\widetilde{G} = \mathrm{GL}(2n, F)$, and put $\sim$ on the previous notation for $\widetilde{G}$.

Notation for the symplectic group :

$G = \mathrm{Sp}(2n, F)$: subgroup of matrices in $\widetilde{G}$ preserving the alternating form b with antidiagonal matrix with coefficient $b(e_i, e_j) = (-1)^{i-1}$ for $i + j = 2n + 1$.

- The centre of G is the group μ of square roots of 1 in F^*.
- $K = G \cap \widetilde{K} = \mathrm{Sp}(2n, \mathcal{O}_F)$.
- $I = G \cap \widetilde{I}$, $I^1 = G \cap \widetilde{I^1}$, $I^2 = G \cap \widetilde{I^2}$, $I^1/I^2 = \kappa^{n+1}$ sending

$$x \mapsto (x_{1,2} \text{ mod. } \mathfrak{p}_F, \ldots, x_{n,n+1} \text{ mod. } \mathfrak{p}_F, x_{2n,1}/\varpi \text{ mod. } \mathfrak{p}_F).$$

Let λ_ψ be the character of I^1 given by ψ on each coordinate of κ^{n+1}.

3) Simple cuspidals for $\mathrm{Sp}(2n)$

Theorem

The intertwining set of λ_ψ in G is $J = \mu I^1$, which is also its normalizer.

Corollary

If λ is any character of J extending λ_ψ, $\mathrm{ind}_J^G \lambda$ is a cuspidal representation of G.

1. Given ψ, λ is determined by its value on μ, given by a sign if p is odd, and trivial if $p = 2$ (then I^1 contains μ).
2. As for $\widetilde{G}$, we may allow different characters on each coordinate of κ^{n+1}. It does not give new cuspidals when $p = 2$, but it gives twice more when p is odd, because κ^* modulo squares has order 2.
3. The cuspidals in 2) are the **simple** cuspidals of G.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

- F^s a separable algebraic closure of F.
- W_F its Weil group.
- Other notation as in section 2.

The local Langlands conjecture (Laumon, Rapoport & Stuhler when $\mathrm{char}(F) = p$, Harris & Taylor, H., Scholze, when $\mathrm{char}(F) = 0$) attaches to a cuspidal for $\mathrm{GL}(n, F)$ (up to isomorphism) an irreducible n-dimensional representation of W_F (up to isomorphism), and conversely.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

Question : *Let $n > 1$. For a simple cuspidal π for $\mathrm{GL}(n, F)$, determined by ϖ, ψ, $\alpha = \lambda(\Pi)$ and the character χ of κ^* yielding the restriction of λ to U_F, can we describe the representation σ of W_F, of dimension n, associated to π ?*

Answer : *yes, but not easy. Bushnell & H. 2013 give an explicit description of the projective representation given by σ, Imai & Tsushima 2015 give a geometric realization of σ.*

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

Great difference if p divides n or not. The extreme cases are $(p, n) = 1$ and $n = p^r$ for some $r > 0$. The general case is a mix of the two.

For $(p, n) = 1$, Π generates a totally ramified tame extension E of F, of degree n ; we can see E in F^s, W_E as an index n subgroup of W_F, and σ is induced from a character of W_E, equivalently of E^*, which sends

$$1 + x \mapsto \psi(x/\varpi \text{ mod. } \mathfrak{p}_E),$$

for $x \in \mathfrak{p}_E$ (note that $\kappa_E = \kappa$), and by α and χ on Π and the Teichmüller lifts of κ^* (up to a slight explicit sign tweak).

The main difficulty is when $n = p^r$, because then σ is **primitive**, very hard to describe !

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

Main information :

LLC preserves L- and ε-factors for pairs. In particular, if Ψ is a non-trivial character of F and μ a character of F^*, we have

$$\varepsilon(\mu\pi, s, \Psi) = \varepsilon(\mu\sigma, s, \Psi).$$

Looking at the exponent of q^{-s}, we obtain

$$\mathrm{Sw}(\sigma) = 1.$$

Also the central character ω_π of π corresponds to $\det(\sigma)$ via class field theory (ω_π is trivial on $1 + \mathfrak{p}_F$, given by χ on $\mathcal{O}_F^*$, by α^n on π).

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

Moreover σ with $\mathrm{Sw}(\sigma) = 1$ is determined, as is π, by few data, $\det(\sigma)$ and $\varepsilon(\mu\sigma, s, \Psi)$ for tame μ. Taking Ψ to be trivial on $\mathfrak{p}_F$ and given by ψ on $\mathcal{O}_F$, then one computes $\varepsilon(\mu\sigma, s, \Psi)$ in terms of $\varpi, \alpha, \chi, \mu$.

When n is prime to p, one checks that the above description gives the answer (Adrian and Liu 2016).

When n is a power of p, $\mathrm{Sw}(\sigma) = 1$ implies that σ is indeed primitive. Let G be the image of σ and G_1 its wild inertia subgroup. By work of H. Koch in the 1970's, G_1 is a Heisenberg type group and there is a minimal Galois tame extension E/F such that the restriction σ_E to E becomes induced from degree p extensions (n^2 of them actually), equivalently is stable by twisting by an order p character.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

The main tools for Bushnell & H. are the theory of base change (Arthur & Clozel) and explicit tame versions (Bushnell & H.).

For a cyclic extension K/F, base change constructs the procedure parallel to restriction to W_K on the Weil group side, but independently of that side.

B. & H. determine E/F using tame base change, and the projective representation attached to σ_E using character twists. They find E/F as an explicit totally ramified extension of degree $n+1$, and explicit equations for the inducing extensions E'/E.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{GL}(n)$.

Imai and Tsushima proceed differently. Motivated by B. & H. and their own work on Deligne-Lusztig varieties, they guess G as a group, and make it act on a curve over κ. Using an Artin-Schreier sheaf attached to ψ, they get a representation of G on its cohomology. Then they produce G as a quotient of W_F and check that the representation thus obtained is the right one.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{Sp}(2n)$

- Notation as in Section 3.
- Hypothesis : char(F)=0.

Arthur associates to a cuspidal (more generally, a discrete series) π for $G = \mathrm{Sp}(2n, F)$ a morphism $\phi = \phi(\pi)$ of $W_F \times \mathrm{SL}(2, \mathbb{C})$ into $\widehat{G} = \mathrm{SO}(2n+1, \mathbb{C})$, up to conjugation by $\widehat{G}$, with the $\widehat{G}$-irreducibility condition that ϕ is the direct sum of inequivalent irreducible orthogonal representations $\phi_1, \ldots, \phi_r$. The discrete series with the same parameter ϕ form an L-packet $L(\phi)$ with 2^{r-1} elements. Given a ≪Whittaker datum≫, the L-packet contains a unique element with a corresponding Whittaker model.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{Sp}(2n)$

Question : *Assume π simple. Can we describe ϕ ?*

1. ϕ is trivial on $\mathrm{SL}(2, \mathbb{C})$. That is easy, because π has a Whittaker model. Indeed, if ϕ is not trivial on $\mathrm{SL}(2, \mathbb{C})$, the element in $L(\phi)$ with a Whittaker model cannot be supercuspidal (Mœglin, Xu).
2. $r = 1$ or 2. In fact, ϕ is either irreducible, or is the sum of a character and an irreducible representation of dimension $2n$ (Oi).That is hard, and uses the full strength of Arthur's construction via endoscopy and twisted endoscopy.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{Sp}(2n)$

Two different outcomes :

- p odd (Oi) : ϕ is reducible (indeed there is no irreducible orthogonal representation of W_F with odd dimension > 1), in fact $r = 2$. The two irreducible components are explicitly determined by the character λ which induces π. One is a character, and the other, of dimension $2n$, is in fact the parameter attached to a simple cuspidal for $\mathrm{GL}(2n, F)$, corresponding to the same choices of ϖ and ψ.
- $p = 2$ (H. for $F = \mathbb{Q}_2$ using results of Adrian & Kaplan, H. & Oi in general) : ϕ is irreducible, and is the representation attached to a simple cuspidal for $\mathrm{GL}(2n+1, F)$, which is explicited from the data defining λ, and also corresponds to the same choice of ϖ and ψ.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{Sp}(2n)$

Remarks :

1. Arthur in fact does not give ϕ directly, but the representations π_i of general linear groups corresponding to the ϕ_i's, and indeed we use his endoscopic and twisted endoscopic character relations to get a hold on the π_i's, not on ϕ directly.
2. When p is odd, Oi also treats split special orthogonal groups. We have now completed the case p=2, using the approach of Adrian, and computations by Adrian and Kaplan.
3. The construction of Gross & Reeder has been generalized by Reeder & Yu to some (tame) non-split groups. When p is odd, Oi treats tamely ramified (non-split) special orthogonal groups, and also unramified unitary groups. When $p = 2$, unramified non-split special orthogonal groups and unramified unitary groups remain to be treated.

4) The Langlands correspondence for simple cuspidals : the case of $\mathrm{Sp}(2n)$

Let us give a bit of (simplified) detail on how Property 2 is proved.

Assume the component ϕ_i has dimension n_i. There is an ≪endoscopic group≫ H of G with L-group (nearly) the product of the $\mathrm{O}(n_i, \mathbb{C})$. Then ϕ factors through the L-group of H, hence a parameter ϕ' for H and a corresponding packet $L(\phi')$. Arthur has shown that, for a regular semisimple element g of G, there is an endoscopic character relation equating a certain linear combination (with signs as coefficients) of the characters at g of the elements of $L(\phi)$ to a linear combination (with ≪transfer factors≫ as coefficients) of the characters of the elements of $L(\phi')$ at the ≪norms≫ of g. Oi selects nice elements g (which he calls ≪affine generic≫) and shows that at some such g the linear combination for G does not vanish. It follows that the linear combination for H does not vanish. But the characteristic polynomial of a norm of a nice g is irreducible (of degree $2n$), which imposes $r = 1$, $n_1 = 2n + 1$, or $r = 2$, $n_1 = 1$ and $n_2 = 2n$.