Dihedral families of GL_n-automorphic L-functions, and equidistribution

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Setup

- Fix $n \geq 2$ an integer. Fix K a CM field, i.e. a totally imaginary quadratic extension of its maximal totally real subfield $K^+ = F$.

- Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$, conjugate self-dual. We shall often assume that π arises as the quadratic basechange lifting $\pi = \mathrm{BC}_{K/F}(\pi')$ of a self-dual cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbf{A}_F)$.

- Let $\chi = \otimes_v \chi_v$ be a ring class character of K.

  $\implies$ there exists an ideal $\mathfrak{c} \subset \mathcal{O}_F$ such that χ factors through the class group of the $\mathcal{O}_F$-order $\mathcal{O}_\mathfrak{c} := \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ of conductor $\mathfrak{c}$ in K:

$$\chi : \mathrm{Pic}(\mathcal{O}_\mathfrak{c}) := \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times \longrightarrow \mathbf{S}^1.$$

- Such characters have trivial restriction to the totally real basefield $\chi_{\mathbf{A}_\mathbf{F}^\times} = 1$, and are equivariant under complex conjugation.

Functional equations

- Consider the standard L-function of π twisted by such a χ: $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty) L(s, \pi \otimes \chi)$.

- Each L-function $\Lambda(s, \pi \otimes \chi)$ has a well-known analytic continuation, and satisfies a *symmetric* functional equation

$$\Lambda(s, \pi \otimes \chi) = \epsilon(s, \pi \otimes \chi) \Lambda(1 - s, \pi \otimes \chi).$$

- Here,

$$\epsilon(s, \pi \otimes \chi) := q(\pi \otimes \chi)^{\frac{1}{2} - s} \epsilon(1/2, \pi \otimes \chi)$$

denotes the epsilon factor with conductor $q(\pi \otimes \chi) \in \mathbf{Z}_{\geq 1}$, and $\epsilon(1/2, \pi \otimes \chi) \in \mathbf{S}^1 \cap \mathbf{R} = \{\pm 1\}$ the root number.

- Observe: If $\epsilon(1/2, \pi \otimes \chi) = -1$, then $\Lambda(1/2, \pi \otimes \chi) = 0$.

Root number dichotomy

- The root number $\epsilon(1/2, \pi \otimes \chi)$ is generically independent of the character, in the sense that there exists $k \in \{0, 1\}$ such that $\epsilon(1/2, \pi \otimes \chi) = (-1)^k$ for all "sufficiently ramified" χ.
- Here, we shall fix a prime $\mathfrak{p} \subset \mathcal{O}_F$, and consider the set $\mathfrak{X}_K(\mathfrak{p})$ of all (primitive) ring class characters χ of K of $\mathfrak{p}$-power conductor.
- In particular, there exists a $k \in \{0, 1\}$ such that $\epsilon(1/2, \pi \otimes \chi) = (-1)^k$ for all by finitely many $\chi \in \mathfrak{X}_K(\mathfrak{p})$.

Dihedral towers

- Again, we fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$. Given an integer $\alpha \geq 0$, consider the corresponding ring class group

$$X(\alpha) := \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha}) := \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^\alpha}^\times$$

- We consider the natural profinite limit $X(\infty) = \varprojlim_\alpha X(\alpha)$.

- Writing $X_0 = X(\infty)_{\mathrm{tors}}$ to denote its finite torsion subgroup, and $\delta_\mathfrak{p} = [F_\mathfrak{p} : \mathbf{Q}_p]$ to denote the residue degree of $\mathfrak{p}$, we have an isomorphism of topological groups $X(\infty) \cong \mathbf{Z}_p^{\delta_\mathfrak{p}} \times X_0$.

A conjecture

- Recall that a ring class character χ of $X(\alpha)$ is *primitive* if it does not factor through $X(\alpha')$ for any $\alpha' < \alpha$.

- Given a character χ_0 of X_0, we write $\mathfrak{X}_K(\alpha; \chi_0)$ to denote the set of primitive ring class characters of conductor $\mathfrak{p}^\alpha$ whose restriction to the finite torsion subgroup X_0 is given by χ_0, i.e. the set of primitive $\chi \in X(\alpha)^\vee$ with $\chi|_{X_0} = \chi_0$.

Conjecture ("Mazur"/folklore)

Fix a character χ_0 of $X_0 = X(\infty)_{\mathrm{tors}}$. The following property true according to the generic root number parametrized by $k \in \{0, 1\}$. For each sufficiently large integer $\alpha \gg 1$, there exists a character $\chi \in \mathfrak{X}_K(\alpha; \chi_0)$ such that $L^{(k)}(1/2, \pi \otimes \chi)$.

- Known by theorems of Cornut-Vatsal – generalizing older theorems of Rohrlich and Greenberg for the dihedral case – for rank $n = 2$.

Motivations, applications

- In the special case of rank $n = 2$, pairing with Iwasawa main conjectures gives bounds for Mordell-Weil ranks for elliptic curves.
- Given an integer $\alpha \geq 0$, let $K[\mathfrak{p}^\alpha]$ denote the ring class extension of conductor $\mathfrak{p}^\alpha$ over K. Hence, $X(\alpha) \cong \mathrm{Gal}(K[\mathfrak{p}^\alpha]/K)$ (by CFT). Let

$$K[\mathfrak{p}^\infty] = \bigcup_{\alpha \geq 0} K[\mathfrak{p}^\alpha]$$

denote the tower of all ring class extensions of K of $\mathfrak{p}$-power conductor.

Theorem (Bertolini-Darmon, Howard et al. + Vatsal/Cornut-Vatsal.)
Let E/F be a modular elliptic curve with cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_F)$, and $\pi = \mathrm{BC}_{K/F}(\pi')$ its lifting to $\mathrm{GL}_2(\mathbf{A}_K)$. Assume E has good ordinary reduction at $\mathfrak{p}$, together with some other technical conditions, and for simplicity that $\delta_\mathfrak{p} := [F_\mathfrak{p} : \mathbf{Q}_p] = 1$. If $k = 0$, then $E(K[\mathfrak{p}^\infty])$ is finitely generated. If $k = 1$, then $E(K[\mathfrak{p}^\alpha]) = [K[\mathfrak{p}^\alpha] : K] + O(1)$ for all sufficiently large $\alpha \gg 1$.

Automorphic periods

- When π is regular algebraic conjugate self-dual (RACSD) cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$, we hope to derive similar implications in higher-rank settings through Iwasawa-Greenberg and Bloch-Kato main conjectures for the corresponding Galois representation – as constructed in the "ten-author" paper by Allen-Calegari-Caraiani-Gee-Helm-Le Hung-Newton-Scholze-Taylor.

- Point of departure in works of Vatsal, Cornut and Cornut-Vatsal:
  - Replace the values $L^{(k)}(1/2, \pi \otimes \chi)$ with period formulae, e.g. Waldspurger/Gross for $k = 0$ and Gross-Zagier for $k = 1$.
  - Reduce to studying Galois or toric orbits of these periods, and to purely group/ergodic theoretic properties.

Two reductions to “toric periods”

- Let us now assume we are in the setting of $k = 0$ with generic root number 1, studying central values $L(1/2, \pi \otimes \chi)$ (as the technology is not yet sufficiently well-developed to study central derivative values).
- Key observation: We have two ways to proceed in this way for the higher-rank setting (the first of which leads to new proofs and arguments in the setting of rank $n = 2$):

(1) Eulerian integral presentations (Hecke/classical for $n = 2$, by Cogdell, Ginzburg, Jacquet, Piatetski-Shapiro, Shalika + Matringe for $n \geq 2$).

(2) Ichino-Ikeda Gan-Gross-Prasad conjectures for $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F)$ (work-in-progress of Beuzart-Plessis and Chaudouard).

(1) Eulerian integral presentations

- Let $\psi = \otimes_v \psi_v$ be the standard additive character of $\mathbf{A}_K/K$, which we extend in the usual way to one of the standard unipotent subgroup $N_n(\mathbf{A}_K) \subset \mathrm{GL}_n(\mathbf{A}_K)$ of upper triangular matrices.
- Recall that given a vector $\varphi \in V_\pi$, we define the corresponding Whittaker coefficient $W_{\varphi,\psi}$ as a function of $g \in \mathrm{GL}_n(\mathbf{A}_K)$ as

$$W_{\varphi,\psi}(g) = \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \varphi(ng)\psi^{-1}(n)dn.$$

- Let $Y_{n,1}$ denote the unipotent radical of the parabolic subgroup attached to the partition $2 + 1 + \cdots 1$ of n, so that $N_n \cong N_2 \ltimes Y_{n,1}$.

$$Y_{n,1}(\mathbf{A}_K) = \left\{ \begin{pmatrix} 1 & 0 & u_{1,3} & u_{1,4} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & u_{2,4} & \dots & u_{2,n} \\ & & 1 & u_{3,4} & \cdots & u_{3,n} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & u_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \quad u_{i,j} \in \mathbf{A}_K \right\}.$$

- Let $\varphi = \otimes_v \varphi_v \in V_\pi$ be a pure tensor whose nonarchimedean local components $\varphi_v, v < \infty$ are "essential Whittaker vectors".
- Let $P_2 \subset \mathrm{GL}_2$ denote the mirabolic subgroup,

$$P_2(\mathbf{A}_K) = \left\{ \begin{pmatrix} y & x \\ & 1 \end{pmatrix}, \quad x \in \mathbf{A}_K, y \in \mathbf{A}_K^\times \right\},$$

- Consider the function defined on $p \in P_2(\mathbf{A}_K)$ by the integral

$$\mathbb{P}_1^n \varphi(p) := |\det(p)|^{-\left(\frac{n-2}{2}\right)} \int_{Y_{n,1}(K) \backslash Y_{n,1}(\mathbf{A}_K)} \varphi\left(u \begin{pmatrix} p & \\ & \mathbf{1}_{n-2} \end{pmatrix} \right) \psi^{-1}(u) du$$

- Remarkable fact: $\mathbb{P}_1^n\varphi$ determines an L^2-automorphic form/function on $P_2(\mathbf{A}_K)$. It is "cuspidal" in the sense that it has the Fourier-Whittaker expansion

$$\mathbb{P}_1^n\varphi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = |y|^{-\left(\frac{n-2}{2}\right)} \sum_{\gamma\in K^\times} W_{\varphi,\psi}\left(\begin{pmatrix} \gamma y & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\psi(\gamma x).$$

- Choosing the pure tensor $\varphi \in V_\pi$ as we do, the Fourier-Whittaker coefficients in the expansion of $\mathbb{P}_1^n\varphi$ have some precise relation to the L-function coefficients of $L(s,\pi)$. In particular, we have relations

$$\Lambda(s,\pi\otimes\chi) = \int_{\mathbf{A}_K^\times/K^\times} \mathbb{P}_1^n\varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)\chi(y)|y|^{s-\frac{1}{2}}dy$$
$$= \int_{\mathbf{A}_K^\times} W_{\varphi,\psi}\left(\begin{pmatrix} y & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(y)|y|^{s-\frac{n-1}{2}}dy.$$

(2) The Ichino-Ikeda Gan-Gross-Prasad conjecture

- Assume now that $n = 2m$ (is even).

- Let V be a hermitian vector space of dimension n over K, with corresponding unitary group $U_n := U(V)$.

- Let $L \subset V$ be a line whose orthogonal complement $L^{\perp} \subset V$ admits an isotropic subspace Z of the maximal possible dimension $m - 1$. Consider the corresponding unitary group $U_1 := U(L)$. Note that there is a natural embedding $U_1 \subset U_n$ induced by the inclusion $L \subset V$, and also that $U_1(F) \backslash U_n(\mathbf{A}_F) \cong \mathbf{A}_K^{\times} / K^{\times}$.

- Let $P \subset U(V)$ be the parabolic subgroup which stabilizes a complete flag of subspaces in Z. Hence P contains $U_1 = U(L)$. Let $N \subset P$ be the unipotent radical.

- Let ψ be an automorphic or additive character on the quotient $N(K) \backslash N(\mathbf{A}_K)$ which is invariant by conjugation by $U_1(\mathbf{A}_K)$, as constructed in Gan-Gross-Prasad.

- Let π^U be a cuspidal automorphic representation of $U_n(\mathbf{A}_F)$.
  - This π^U sometimes admits a basechange to a cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbf{A}_F)$, which in turn has a quadratic basechange $\pi = \mathrm{BC}_{K/F}(\pi')$ to $\mathrm{GL}_n(\mathbf{A}_K)$ as we consider above.
  - In this setting, we have an identification of completed L-functions $\Lambda(s, \pi \otimes \chi) = \Lambda(s, \pi^U \otimes \chi)$ for any ring class character χ of K.
- In this setting, define an associated projection operator $\mathcal{P}_\psi$: for $t \in U_1(\mathbf{A}_F) \cong \mathbf{A}_K^\times$ and $\phi = \otimes_v \phi_v \in V_{\pi^U}$ a decomposable vector,

$$\mathcal{P}_\psi \phi(t) := \int_{N(K)\backslash N(\mathbf{A}_K)} \phi(ut)\psi^{-1}(t)dt.$$

- Let χ be a ring class character of K, which we now identify as an automorphic character of $U_1(F)\backslash U_n(\mathbf{A}_F)$.

- Consider the linear form defined by

$$\phi \in \pi^U \longmapsto P_\chi(\phi) := \int_{U_1(F)\backslash U_1(\mathbf{A}_F)} \mathcal{P}_\psi \phi(t) \cdot \chi(t) dt$$

Conjecture (Ichino-Ikeda conjecture for $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F)$)

There exists for each place v of K a local sesqui-linear form $P_{\chi_v} : \pi_v^U \times \pi_v^U \to \mathbf{C}$ such that for any decomposable vector $\phi = \otimes_v \phi_v \in \pi^U$, we have the identification

$$|P_\chi(\phi)|^2 \approx \frac{\Lambda(1/2, \pi^U \otimes \chi)}{\Lambda(1, \pi^U, \mathrm{Ad})} \cdot \prod_v P_{\chi_v}(\phi_v, \phi_v).$$

Here, $\Lambda(s, \pi^U, \mathrm{Ad}) = \Lambda(s, \pi, \mathrm{Ad})$ denotes adjoint L-function of π^U, and the $\approx$ means given up to special values of abelian L-functions which cancel out to one given suitable choices of Haar measures.

A generalization of the Vatsal/Cornut-Vatsal approach

Conjecture (Refinement for $k = 0$)

Assume we are in the setup of generic root number $+1$ parametrized by $k = 0$ in the setup outlined above. Fix a character χ_0 of the finite torsion subgroup $X_0 = X(\infty)_{\mathrm{tors}}$. There exists for each sufficiently large integer $\alpha \gg 1$ a primitive ring class character $\chi \in \mathfrak{X}_K(\alpha; \chi_0)$ such that the following equivalent conditions hold:

(1) *The Eulerian integral*

$$\int_{\mathbf{A}_K^\times / K^\times} \mathbb{P}_1^n \varphi \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \right) \chi(y) dy = \int_{\mathbf{A}_K^\times} W_{\varphi,\psi} \left(\begin{pmatrix} y & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) \chi(y) |y|^{-\frac{n-2}{2}} dy$$

does not vanish.

(2) *The automorphic period $P_\chi(\phi)$ does not vanish.*

Fourier analytic setup

- Given a class $A \in X(\alpha)$ (for any $\alpha \geq 0$) with some idele representative $t \in \mathbf{A}_K^\times$ (so that $A = [t]$), define functions

  (1) $\mathfrak{W}_\varphi(A) = \mathfrak{W}_\varphi([t]) = \sum_{\substack{\lambda \in tK^\times K_\infty^\times \widehat{\mathcal{O}}_{\mathfrak{p}^\alpha}^\times \\ [t] = A \in X(\alpha)}} \mathbb{P}_1^n \varphi \left(\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} \right)$

  (2) $\mathcal{W}_\phi(A) = \mathcal{W}_\phi([t]) = \sum_{\substack{\lambda \in tK^\times K_\infty^\times \widehat{\mathcal{O}}_{\mathfrak{p}^\alpha}^\times \\ [t] = A \in X(\alpha)}} \mathcal{P}_\psi \phi(\lambda).$

- It is enough to study the finite sums

  (1) $\mathbf{a}(\alpha, \chi) := \# X(\alpha)^{-1} \sum_{A \in X(\alpha)} \mathfrak{W}_\varphi(A) \chi(A)$

  (2) $\mathbf{a}(\alpha, \chi) := \# X(\alpha)^{-1} \sum_{A \in X(\alpha)} \mathcal{W}_\phi(A) \chi(A).$

- It is enough to estimate the weighted averages $\mathbf{b}(\alpha, \chi_0) := \# \mathfrak{X}_K(\alpha; \chi_0)^{-1} \cdot \sum_{\chi \in \mathfrak{X}_K(\alpha; \chi_0)} \mathbf{a}(\alpha, \chi).$

Strategy

- Reduce to a certain equidistribution criterion.
- Reduce the criterion to one about $\mathfrak{p}$-adic unipotent flows.
- Deduce the corresponding criterion about $\mathfrak{p}$-adic unipotent flows from deep (general) theorems of Ratner and Margulis-Tomanov.

Group actions setup

- Evaluating $\mathbf{b}(\alpha, \chi_0)$ using the corresponding orthogonality relation (derived via the inclusion-exclusion principle), reduce to considering sums over the finite torsion subgroup $X_0 = X(\infty)_{\text{tors}}$.

- Motivated secretly by existing deep theorems on p-adic unipotent flows, consider the following filtration $X_0 \supset X_1 \supset X_2 \supset \{1\}$ of this finite torsion subgroup: X_1 is the torsion subgroup corresponding the dense but countable subgroup of $X(\infty)$ generated by uniformizers away from $\mathfrak{p}$; $X_2 \cong \text{Pic}(\mathcal{O}_F)$ corresponds to the class group of F (but can be ignored for most of the subsequent arguments).

- Fix a set of representatives $\mathcal{R}$ of X_0/X_1.

Proposition

It is enough to show that for (1) some explicit function φ'' constructed from $\varphi \in V_\pi$ or (2) some explicit function ϕ'' constructed from $\phi \in V_{\pi^U}$,

$$\mathbf{b}(\chi_0) := \sum_{\sigma=[\tau]\in\mathcal{R}} \chi_0(\sigma)\mathfrak{W}_{\varphi''}(\sigma) \neq 0 \qquad \textit{via (1)}$$

$$\mathbf{b}^U(\chi_0) := \sum_{\sigma=[\tau]\in\mathcal{R}} \chi_0(\sigma)\mathcal{W}_{\phi''}(\sigma) \neq 0 \qquad \textit{via (2).}$$

- Idea (going back to Vatsal for $n = 2$): Use some ergodic/group theoretic property to handle these remaining sums.

- Let us for simplicity restrict to the setup (1); the corresponding setup for (2) is a simple adaptation to unitary groups.
- Fix a compact open subgroup $H \subset \mathrm{GL}_n(\mathbf{A}_{K,f})$.
- Consider the embedding

$$\mathbf{A}_K^\times \longrightarrow \mathrm{GL}_n(\mathbf{A}_K), \quad t \longmapsto \begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix}.$$

- Consider the corresponding space of toric points

$$T_H := K^\times \backslash \mathrm{GL}_n(\mathbf{A}_{K,f})/H$$

the space of special points

$$M_H := \mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbf{A}_{K,f})/H,$$

and the space of connected components

$$N_H := Z\left(\mathrm{GL}_n(K)\right) \backslash Z\left(\mathrm{GL}_n(\mathbf{A}_{K,f})\right) / \det(H) \cong K^\times \backslash \mathbf{A}_{K,f}^\times / \det(H).$$

- There is a natural action of $\mathbf{A}_{K,f}^{\times}$ on each of these spaces by left multiplication; i.e. via the embedding defined above. We sometimes write $\star$ to denote this action.

- These spaces are connected by a natural reduction maps

$$\mathrm{red} : T_H \longrightarrow M_H \quad \text{and} \quad c : M_H \longrightarrow N_H,$$

- The composition

$$c \circ \mathrm{red} : T_H \longrightarrow M_H \longrightarrow N_H.$$

  is $\mathbf{A}_K^{\times}$-equivariant.

Simultaneous reduction maps

- Consider the simultaneous reduction maps

$$T_H \xrightarrow{\text{RED}} M_H^{\mathcal{R}}, \quad g \longmapsto (\text{red}(\tau \cdot g))_{\tau \in \mathcal{R}} = (a_\tau)_{\tau \in \mathcal{R}}$$

$$M_H^{\mathcal{R}} \xrightarrow{C} N_H^{\mathcal{R}}, \quad (a_\tau)_{\tau \in \mathcal{R}} \longmapsto (c(a_\tau))_{\tau \in \mathcal{R}}.$$

- Consider the composition of these maps

$$T_H \xrightarrow{\text{RED}} M_H^{\mathcal{R}} \xrightarrow{C} N_H^{\mathcal{R}}, g \longmapsto \overline{g} := C \circ \text{RED}(g) = (c(a_\tau))_{\tau \in \mathcal{R}}.$$

Conjecture ("surjectivity")

For all but finitely many toric points $g \in T_H$ (of "$\mathfrak{p}$-power conductor"),

$$\mathrm{RED}(X \star g) = C^{-1}\left(X \star C \circ \mathrm{RED}(g)\right).$$

Proposition

If this latter conjecture is true, then the (simplified, reformulated) nonvanishing conjecture stated above follows as a formal consequence.

- As in Vatsal/Cornut-Vatsal, the proof follows from a simple contradiction argument, taking "surjectivity" for granted.

Equidistribution in disguise

- Fix a prime $\mathfrak{P}$ above $\mathfrak{p}$ in K.

- Define a $\mathfrak{P}$-*isogeny class* $\mathcal{H}$ to be a $\mathrm{GL}_n(K_{\mathfrak{P}})$-orbit in T_H.

Conjecture ("equidistribution")

Let $\mathcal{H} \subset T_H$ be any $\mathfrak{P}$-isogeny class, and $\mathcal{X} \subset \mathbf{A}_{K,f}^{\times}$ any compact subset with Haar measure dx. Then for each continuous function $f : M_H^{\mathcal{R}} \to \mathbf{C}$, the function

$$g \longmapsto \int_{\mathcal{X}} f \circ \mathrm{RED}\,(x \cdot g)\, dx - \int_{\mathcal{X}} dx \int_{C^{-1}(x \cdot \overline{g})} f d\mu_{x \cdot \overline{g}}$$

converges to 0 *as g goes to infinity in $\mathcal{H}$. Here, $d\mu_z$ denotes the Borel probability measure on $N_H^{\mathcal{R}}$, and we write $\overline{g} = C \circ \mathrm{RED}(g)$.*

- Easy check: "equidistribution" $\implies$ "surjectivity".

Reductions to p-adic unipotent flows

- Remaining strategy: (I) reduce “equidistribution” to a statement about $\mathfrak{p}$-adic unipotent flows. (II) deduce what is needed from the theorems of Ratner and Margulis-Tomanov.

- For (I), we expect or conjecture that out “equidistribution” conjecture is implied by the following claim.

Claim (I)

The following assertions are true.

(i) *Fix a Haar measure* $d\nu$ *on* $K_{\mathfrak{P}}$, *as well as a toric point* $g \in T_H$. *Let* $\mathcal{N} = \{n(t) : t \in K_{\mathfrak{P}}\}$ *be any one-parameter unipotent subgroup of* $\mathrm{SL}_n(K_{\mathfrak{P}})$. *Let us for any integer* $\alpha \geq 0$ *write* $\mathfrak{K}_\alpha = \varpi_{\mathfrak{P}}^{-\alpha}\mathfrak{K}$ *for* $\mathfrak{K} := 1 + \varpi_{\mathfrak{P}}\mathcal{O}_{K_{\mathfrak{P}}}$ *(with* $\varpi_{\mathfrak{P}}$ *a fixed uniformizer of* $\mathfrak{P}$*), so that* $\nu(\mathfrak{K}_\alpha) \to \infty$ *as* $\alpha \to \infty$. *If we have a dense subset inclusion*

$$\mathrm{RED}(g \cdot \mathcal{N}) \subset C^{-1}(\overline{g}) = C^{-1}(C \circ \mathrm{RED}(g)),$$

then we have $\mathfrak{p}$*-adic equidistribution:*

$$\lim_{\alpha\to\infty} \frac{1}{\nu(\mathfrak{K}_\alpha)} \int_{\mathfrak{K}_\alpha} \mathrm{RED}(g \cdot n(t))dt = \int_{C^{-1}(\overline{g})} d\mu_{\overline{g}}.$$

(ii) *For all but finitely many* $\sigma \in \mathcal{X} = X = \varprojlim_{\alpha\to\infty} X_\alpha$, *we have the (desired) dense subset inclusion* $\mathrm{RED}(\sigma \cdot g \cdot \mathcal{N}) \subset C^{-1}(\sigma \cdot \overline{g})$.

- We expect Claim (I) can be deduced from the theorems of Ratner and Margulis-Tomanov on p-adic unipotent flows. Sketch of setup:
  - Put $r = \#\mathcal{R}$.
  - Argue that it is enough to consider locally constant functions f.
  - Any locally constant function $f : C^{-1}(\overline{g}) \longrightarrow \mathbf{C}$ factors through $C^{-1}(\overline{g})/H^r = \mathrm{RED}/H^r$.
  - There is a natural action of $\mathrm{SL}_n(K_{\mathfrak{P}})^r \subset \mathrm{GL}_n(K_{\mathfrak{P}})^r$ on $C^{-1}(\overline{g})$.
  - We consider the corresponding stabilizer subgroups

$$\Gamma(g,H) = \mathrm{Stab}_{\mathrm{SL}_n(K_{\mathfrak{P}})^r}(\mathrm{RED}(g)\cdot H^r) \cong \prod_{\tau\in\mathcal{R}} \mathrm{Stab}_{\mathrm{SL}_n(K_{\mathfrak{P}})}(\mathrm{red}(\tau\cdot g)\cdot H).$$

  - We claim there is an $\mathrm{SL}_n(K_{\mathfrak{P}})^r$-invariant homeomorphism

$$\Gamma(g,H)\backslash \mathrm{SL}_n(K_{\mathfrak{P}})^r \longrightarrow C^{-1}(\overline{g})/H^r, \quad (\gamma_\tau)_{\tau\in\mathcal{R}} \longmapsto \mathrm{RED}(g)\cdot(\gamma_\tau)_{\tau\in\mathcal{R}}.$$

  - The map $t \longmapsto \mathrm{RED}(g\cdot n(t))$ on $C^{-1}(\overline{g})/H^r$ corresponds under this homeomorphism to the image of $t \longmapsto \Delta\circ n(t)$ on $\Gamma(g,H)\backslash \mathrm{SL}_n(K_{\mathfrak{P}})^r$.

Relate Claim (I) in this setup to a more standard-looking statement:

Claim (II)

Let $\mathcal{N} = \{n(t) : t \in K_{\mathfrak{P}}\}$ *any one-parameter unipotent subgroup in* $\mathrm{SL}_n(K_{\mathfrak{P}})$. *If* $\Gamma(g, H) \subset \mathrm{SL}_n(K_{\mathfrak{P}})^r$ *is dense, then for any continuous function* $f : \Gamma(g, H)\backslash \mathrm{SL}_n(K_{\mathfrak{P}})^r \longrightarrow \mathbf{C}$, *we have that*

$$\lim_{m\to\infty} \frac{1}{\nu(\mathfrak{K}_m)} \int_{\mathfrak{K}_m} f\left(\Delta \circ n(t)\right) dt = \int_{\Gamma(g,H)\backslash \mathrm{SL}_n(K_{\mathfrak{P}})^r} f d\mu_{\Gamma(g,H)}.$$

- Given a subgroup $\Gamma \subset \mathrm{SL}_n(K_{\mathfrak{P}})$, let $[\Gamma]$ to denote its commensurability class, i.e. the collection of subgroups $\Gamma' \subset \mathrm{SL}_n(K_{\mathfrak{P}})$ for which $\Gamma \cap \Gamma'$ has finite index in each of Γ and Γ'.
- We expect that for each distinct pair of representatives $\tau, \tau' \in \mathcal{R}$,

$$\mathfrak{B}_g(\tau, \tau') := \left\{ \sigma \in \mathbf{A}_{K,f}^{\times}/K^{\times} : [\Gamma_{\tau}(\sigma \cdot g, H)] \cdot \mathcal{N} = [\Gamma_{\tau'}(\sigma \cdot g, H)] \cdot \mathcal{N} \right\}$$

  is a disjoint union of countably many cosets of $\mathbf{A}_{K,f}^{\mathfrak{P}\times} K^{\times}$ in $\mathbf{A}_{K,f}^{\times} K^{\times}$.
- We also expect that if $\sigma \in \mathbf{A}_{K,f}^{\times}$ is not contained in the "bad" set

$$\mathfrak{B} = \mathfrak{B}_g = \bigcup_{\substack{\tau, \tau' \in \mathcal{R} \\ \tau \neq \tau'}} \mathfrak{B}_g(\tau, \tau'),$$

  then $\mathrm{RED}\,(\sigma \cdot g \cdot \mathcal{N}) \subset C^{-1}\,(\sigma \cdot \overline{g})$ is dense. Moreover, we expect that we can derive an unconditional density criterion/statement here from the theorems of Ratner & Margulis-Tomanov.

The theorems of Ratner and Margulis-Tomanov

- Let $\mathcal{Y} = \Gamma(g, H) \backslash \mathrm{SL}_n(K_{\mathfrak{P}})^r$.
- More generally, we can consider $\mathcal{Y} = \left(\prod_{i=1}^r \Gamma_i\right) \backslash \mathrm{SL}_n(K_{\mathfrak{P}})^r$ for $\prod_{i=1}^r \Gamma_i$ any product of cocompact lattices $\Gamma_i \in \mathrm{SL}_n(K_{\mathfrak{P}})$.
- Let $\mathcal{V} = \{v(t) : t \in K_{\mathfrak{P}}\}$ be any one-parameter unipotent subgroup of $\mathrm{SL}_n(K_{\mathfrak{P}})^r$.
- Given any point $z = (z_i)_{i=1}^r \in \mathcal{Y}$, the main theorems of Ratner and Margulis-Tomanov on p-adic unipotent flows then give us the following:
  - A closed subgroup $L \subseteq \mathcal{V}$ of $\mathrm{SL}_n(K_{\mathfrak{P}})^r$ such that $\overline{z \cdot L} = z \cdot \mathcal{V}$,
  - A unique L-invariant Borel probability measure μ on $\mathcal{Y}$ (determined by z and $\mathcal{V}$) supported on $\overline{z \cdot L}$,
  - For each continuous function $f : \mathcal{Y} \to \mathbf{C}$ and compact subset of positive measure $\mathfrak{K} \subset K_{\mathfrak{P}}$, the uniform distribution property

$$\lim_{|m| \to \infty} \frac{1}{\nu(m \cdot \mathfrak{K})} \int_{m \cdot \mathfrak{K}} f\left(z \cdot v(t)\right) d\nu(t) = \int_{\mathcal{Y}} f(y) d\mu(y).$$

- Clarifying and refining details for these final reductions (to deduce some form of the conjecture this way) is a work in progress ...

¡MUCHAS GRACIAS POR SU ATENCIÓN!

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