Around local and global Langlands correspondences for function fields

Report on joint work with Gan and Sawin; with Böckle, Feng, Khare, and Thorne; and with Ciubotaru

November 30, 2022

What is the local Langlands conjecture?

The surprise answer is that there is no local Langlands conjecture.

Or else, (what comes to the same thing) there are several local Langlands conjectures.

In what follows G is a connected reductive group over a local non-archimedean field F of with residue field $k = \mathbb{F}_q$ of characteristic p; $\Pi(G/F)$ denotes the set of irreducible admissible representations of $G(F)$ with coefficients in C, an algebraically closed field of characteristic zero, with a chosen $q^{\frac{1}{2}}$.

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First version of LLC

A Langlands parameter (L-parameter) for G/F is a pair

$$\rho : W_F \to {}^LG(C), N \in Lie(\hat{G})$$

with W_F the Weil group of F, satisfying the usual relations. The set of equivalence classes of L-parameters for G/F is denoted $\Phi(G/F)$.

A semisimple Langlands parameter for G/F is a homomorphism

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with the property that if $\rho(W_F) \subset P$ for some proper parabolic $P \subset {}^LG(C)$ then the image is contained in a Levi subgroup. The set of equivalence classes of semisimple L-parameters for G/F is denoted $\Phi(G/F)^{ss}$.

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Conjecture

(a) *There is a canonical parametrization*

$$\mathcal{L} = \mathcal{L}_{G/F} : \Pi(G/F) \to \Phi(G/F).$$

(b) *For any $\varphi \in \Phi(G/F)$, the* ***L*-packet** $\Pi_\varphi := \mathcal{L}^\varphi$ *is finite.*

(c) *For any $\varphi \in \Phi(G/F)$ the L-packet Π_φ is non-empty.*

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Some conditions

Property

When G is a torus, the map $\mathcal{L}$ is given by class field theory.

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Compatibility with isomorphisms of fields, twists, contragredients, central isogenies, products, etc.

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Automorphic conditions

(i) Suppose $\sigma : {}^L G \to GL(N)$ is an algebraic representation. Suppose there is a theory of automorphic L-functions for G over F and σ (e.g. Rankin-Selberg or Langlands-Shahidi). Then the L and ε factors are compatible with respect to $\mathcal{L}$.

(ii) Compatibility with (twisted) endoscopic transfer, in particular with cyclic base change.

(iii) Local global compatibility (see below).

Point (i) (for Rankin-Selberg for $GL(n) \times GL(m)$), plus some of the earlier conditions, suffice to characterize LLC for $GL(n)$ (Henniart).
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Current results

But the original conditions are unknown for general G:

(a) **Parametrization:** There is a canonical parametrization

$$\mathcal{L} = \mathcal{L}_{G/F} : \Pi(G/F) \to \Phi(G/F).$$

(b) **Finiteness of L-packets:** for any $\varphi \in \Phi(G/F)$, the ***L*-packet** $\Pi_\varphi := \mathcal{L}^\varphi$ is finite.

(c) **Surjectivity:** any $\varphi \in \Phi(G/F)$ the L-packet Π_φ is non-empty.

Much remains to be done

Parametrization:

A semisimple version of (a)

$$\mathcal{L}^{ss} : \Pi(G/F) \to \Phi(G/F)^{ss}$$

have been constructed by Genestier-Lafforgue (for char $F > 0$) and by Fargues-Scholze (for all F).

For global function fields $\mathcal{L}^{ss}$ is compatible with the global parametrization of Lafforgue and C. Xue (Genestier-Lafforgue for their $\mathcal{L}^{ss}$; Li Huerta for the Fargues-Scholze $\mathcal{L}^{ss}$.

(b) **Finiteness of L-packets** and (c) **Surjectivity** are known (in principle) for classical groups and (probably) G_2 (with the purely local $\mathcal{L}^{ss}$); but not in general.

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Kaletha's parametrization

Kaletha has used the construction of J.-K. Yu to define a partial parametrization $\mathcal{L}$ whose domain is the set of regular and non-singular supercuspidals representations of any $G(F)$. This includes all supercuspidals if p is not too small for G (Fintzen).

Fintzen, Kaletha, and Spice proved character formulas that imply compatibility with endoscopic transfer and characterize the parametrization uniquely (except for small p).

Question: Is Kaletha's parametrization the same as Fargues-Scholze or Genestier-Lafforgue?

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The Deligne-Kazhdan correspondence

Let $F = k((t))$ be a local field of characteristic p and $F^\sharp$ a p-adic field that is *n-close* to F for some $n >> 0$:

$$\mathcal{O}_F/m_F^n \xrightarrow{\sim} \mathcal{O}_{F^\sharp}/m_{F^\sharp}^n$$

Then (Deligne) letting Φ^n denote Weil group parameters of depth n,

$$\Phi^n(G/F) \xrightarrow{\sim} \Phi^n(G/F^\sharp)$$

Let G be split connected, $K_r(F) \subset G(\mathcal{O}_F)$ (resp. $K_r(F^\sharp) \subset G(\mathcal{O}_{F^\sharp})$) the standard m_F^r-congruence subgroup.
Let $H(G(?), r) = H(G(?), K_r(?)), ? = F, F^\sharp$. Then (Ganapathy, Kazhdan) if $F^\sharp$ and F are r-close then letting Π^r denote irreducible representations of depth r.

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An exercise

Conjecture (GHS)

If $n >> r$ the following diagram commutes:

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Application of potential automorphy

Let G be split, $\varphi \in \Phi(G/F)$ irreducible.
(1) Globalize φ: find a global field K with a place v such that $K_v \xrightarrow{\sim} F$ a finite field $\mathbb{F}$ of large characteristic, and a surjective

$$\bar{\rho} : Gal(K^{sep}/K) \to \hat{G}(\mathbb{F})$$

such that $\bar{\rho}_v \xrightarrow{\sim} \varphi$.
(2) Lift $\bar{\rho}$ to $\rho : Gal(K^{sep}/K) \to \hat{G}(W(\mathbb{F}))$ with Zariski dense image.
(3) (Potential automorphy) Prove that $\rho' = \rho|_{Gal(K^{sep}/K')}$ is automorphic for some Galois K'/K, $\rho' = \mathcal{L}(\Pi')$
(4) Descend Π' to an automorphic representation Π_L of G over the fixed field $L \subset K'$ of the decomposition group D_v, which is solvable. Then $\varphi = \mathcal{L}(\Pi_{L,v})$.

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Theorem for G_2

This strategy has actually been applied in one case!

Theorem (H-Khare-Thorne)

For F p-adic and $G = G_2$, this strategy yields a bijection

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Explanation

The construction of $\mathcal{L}_0^{generic}$ is based on the exceptional theta correspondence with $PGSp(6)$ and a mass of work of Ginzburg, Rallis, Soudry, Li, Jiang, Savin, Gan, and others.

Steps (1)-(3) (globalization, lifting, potential automorphy) are essentially as in our paper with Böckle over function fields.

Step (4) (cyclic descent) works because we actually work over $GL(7)$, where descent is available thanks to Arthur-Clozel, as in the proof of LLC for $GL(n)$.

Extra step (*deus ex machina*): We get back from $GL(7)$ to G_2 using an argument based on poles of L-functions due to Hundley-Liu. This probably works over function fields of characteristic > 3.

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Why this may not work in general

Descent means: we take $K' = K_1 \supset K_2 \supset \cdots \supset K_r = L$ where K_i/K_{i+1} is cyclic of prime order.

Descent for the cyclic extension K_i/K_{i+1} means identifying a stable twisted trace for $G(\mathbf{A}_{K_i})$ (action of $Gal(K_i/K_{i+1})$ with a stable trace for $G(\mathbf{A}_{K_{i+1}})$.

But if we don't know multiplicity one for G/K_i, the twisted trace can vanish. In particular, if there are infinitely many supercuspidals π_α with the same $\mathcal{L}(\pi_\alpha)$, descent may fail.

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Descent means: we take $K' = K_1 \supset K_2 \supset \cdots \supset K_r = L$ where K_i/K_{i+1} is cyclic of prime order.

Descent for the cyclic extension K_i/K_{i+1} means identifying a stable twisted trace for $G(\mathbf{A}_{K_i})$ (action of $Gal(K_i/K_{i+1})$ with a stable trace for $G(\mathbf{A}_{K_{i+1}})$.

But if we don't know multiplicity one for G/K_i, the twisted trace can vanish. In particular, if there are infinitely many supercuspidals π_α with the same $\mathcal{L}(\pi_\alpha)$, descent may fail.

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Review of V. Lafforgue's global results

Let Y be a smooth projective curve over $k = \mathbb{F}_q$, $\ell \neq p$ a prime.
G split semisimple over $K = k(Y)$.
$\mathcal{A}_0(G) = \{$ cuspidal automorphic representations of $G(\mathbf{A}_K)\}$,
$\mathcal{G}^{ss}(G) = \{$ semisimple maps $\rho_\ell : Gal(K^{sep}/K) \to \hat{G}(\overline{\mathbb{Q}}_\ell).\}$

Theorem (VL)

There is a map

$$\mathcal{L} : \mathcal{A}_0(G) \to \mathcal{G}^{ss}(G)$$

with the following property: if v is a place of K and $\Pi \in \mathcal{A}_0(G)$ is a cuspidal automorphic representation such that Π_v is unramified, then $\mathcal{L}(\Pi)$ is unramified at v, and the semisimplification $\mathcal{L}^{ss}(\Pi) \mid_{W_{K_v}}$ is the Satake parameter of Π_v.

Local parameters

Theorem (Genestier-Lafforgue, Li Huertax)

With the above hypotheses, let w be any place of K. Then

$$\mathcal{L}_w(\Pi_w) := [\mathcal{L}(\Pi) \mid_{W_{K_w}}]^{ss}$$

depends only on K_w and Π_w (not on the globalizations K and Π). Moreover, $\mathcal{L}_w$ is compatible with parabolic induction in the obvious sense.

In particular, if $F = k((t))$ is an equal characteristic local field and π is an irreducible representation of $G(F)$, we can define the semisimple homomorphism

$$\mathcal{L}(\pi) : W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell).$$

Weights

Let $\sigma : \hat{G} \to GL(N)$ be any representation, S the set of primes where Π is ramified. Then

$$\mathcal{L}(\Pi)_\sigma := \sigma \circ \mathcal{L}(\Pi) : Gal(K^{sep}/K) \to GL(N, \overline{\mathbb{Q}}_\ell)$$

corresponds to a semi-simple ℓ-adic local system $L(\Pi)_\sigma$ on $Y \setminus |S|$.

By Deligne's Weil II, each irreducible summand of $L(\Pi)_\sigma$ is punctually pure (up to twist by a character of $Gal(\bar{k}/k)$).
Hence for any w, the eigenvalues of $\sigma \circ \mathcal{L}_w(\Pi_w)(Frob_w)$ are Weil q-numbers of various weights (up to the twist, which we ignore).

Say a representation π of $G(F)$ is *pure* if for some (equivalently, for any) faithful σ, all the eigenvalues of $\sigma \circ \mathcal{L}(\pi)(Frob_q)$ have the same weight.

What about supercuspidals?

If $G \neq GL(n)$, not all supercuspidals are pure. Here is our main theorem.

Theorem (GHS)

Let π be a **pure** *supercuspidal representation of $G(F)$. Suppose π is compactly induced from a compact open subgroup of G. (For example, if p does not divide the order of the Weyl group $W(G)$, this follows from* **Fintzen's theorem***.) Suppose moreover that $q > 3$. Then $\mathcal{L}(\pi)$ is not unramified.*

Recall that G is split semisimple. But the arguments should work for quasi-split G as well, and the results on base change should probably eliminate the restriction for $q = 3$.

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Unramified pure representations are principal series

Henceforward we assume $p \nmid |W(G)|$, in order to apply Fintzen's theorem. Because $\mathcal{L}$ is compatible with parabolic induction, we conclude

Corollary

Let π be a pure representation of $G(F)$. Suppose $\mathcal{L}(\pi)$ is unramified. Then π is an irreducible constituent of an unramified principal series.

In particular, there are *not* infinitely many supercuspidals with trivial parameter.

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Incorrigible representations

Assume local and global *cyclic stable base change* over function fields works as in Labesse's book. An *incorrigible representation* of $G(F)$ is a supercuspidal representation π such that, for any sequence $F \subset F_1 \subset \cdots \subset F_r$ of cyclic Galois extensions, the base change of π to F_r (which is an L-packet) contains a supercuspidal member.

Corollary (GHS)

No pure supercuspidal representation is incorrigible.

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Two proofs for $G = GL(n)$: Henniart (numerical correspondence), and Scholze (nearby cycles; see also Li Huerta in positive characteristic). Along with the existence of a canonical parametrization, this is **the** key step in any proof of LLC for $GL(n)$. The above corollary is again the key step.

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Wild ramification

Theorem

Suppose π is a pure supercuspidal compactly induced from an open compact subgroup that is **sufficiently small**. *Then $\mathcal{L}(\pi)$ is wildly ramified.*

Example: any subgroup of the principal congruence subgroup $G(\mathcal{O}_F)_+ \subset G(\mathcal{O}_F)$ is sufficiently small.

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Mixed supercuspidals

The supercuspidals π that are not pure have Langlands parameters (ρ, N) with $N \neq 0$. Then we expect that the pair (ρ, N) satisfies **purity of the monodromy weight filtration**: (up to unramified twist)

(i) For any if $\sigma : \hat{G} \to GL(M)$, the eigenvalues of $\sigma \circ \rho(Frob_F)$ are all q-numbers of integer weight.

(ii) The subspace $W_a V \subset V$ of eigenvectors for $\sigma \circ \rho(Frob_F)$ with eigenvalues of weight $\leq a$ is invariant under (ρ, N);

(iii) Letting $gr_a V = W_a V / W_{a-1} V$, there is $w \in \mathbb{Z}$ such that, for all $i \geq 0$, the map

$$N^i : gr_{w-i} V \to gr_{w+i} V$$

is an isomorphism.

Mixed supercuspidals

The parameter $\mathcal{L}^{ss}(\pi)$ (of Genestier-Lafforgue or Fargues-Scholze) does not see N. However, not every semisimple parameter can be completed to one satisfying purity of MWF.
For example, it $\sigma \circ \rho$ has two weights a and a', and if it ρ can be completed to (ρ, N) satisfying purity, then $a \equiv a' \pmod 2$. We prove:

Theorem

For any supercuspidal π, $\mathcal{L}^{ss}(\pi)$ can be completed to a pair $(\mathcal{L}^{ss}(\pi), N)$ that satisfies purity of the monodromy weight filtration.

We are thus entitled to define this pair to be the full Langlands parameter $\mathcal{L}(\pi)$ of π.

Globalization

Let $Y = \mathbb{P}^1$, $K = k(t)$. Choose a Borel $B \subset G$ (over k), B_- an opposite Borel. Let $I_0 \subset G(K_0)$ (resp. $I_{\infty,+} \subset G(K_\infty)$ denote the Iwahori corresponding to B (resp. the pro-unipotent radical of the Iwahori corresponding to B_-). We construct a cuspidal automorphic representation Π of $G(\mathbf{A}_K)$ such that

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k) \subset \mathbb{P}^1(\bar{k})$, $\Pi_z \xrightarrow{\sim} \pi$;

(b) For $x \notin |\mathbb{P}^1(k)|$, Π_x is unramified

(c) $\Pi_\infty^{I_{\infty,+}} \neq 0$.

(d) $\Pi_0^{I_{0,+}}$ contains a vector transforming under a certain character χ_k of $I_0/I_{0,+}$.

Here $I_{0,+} \subset I_0$ is what you think it is.

Application of purity

Suppose $\mathcal{L}(\pi)$ is unramified. By the purity hypothesis, for any faithful $\sigma \in Rep(\hat{G})$, $\mathcal{L}(\Pi)_\sigma$, which is a priori an ℓ-adic local system on $\mathbb{P}^1 \setminus |\mathbb{P}^1(k)|$, extends to a punctually pure local system on $\mathbb{G}_m$. (There is no unipotent monodromy at the points in $\mathbb{G}_m(k)$.) Moreover, our hypotheses imply that the ramification at 0 and ∞ is *tame*. Thus it is a sum of local systems induced from characters of finite order of the tame fundamental group of $\mathbb{G}_m$. Of course, $\mathcal{L}(\Pi)_z = \mathcal{L}(\pi)$ for every $z \in \mathbb{G}_m(k)$. By varying the character χ_k, we obtain a contradiction.

Poincaré series

Suppose for simplicity π is compactly induced from $U = G(\mathcal{O}_F)$. Let φ_π be a matrix coefficient of π supported in U, $\varphi_\pi(1) = 1$.

We construct Poincaré series on $G(\mathbf{A}_K)$ as in the Gan-Lomelí paper

$$P_\varphi(g) = \sum_{\gamma \in G(K)} \varphi(\gamma \cdot g), g \in G(\mathbf{A}_K)$$

where $\varphi = \prod_x \varphi_x$ with

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k)$, $\varphi_z = \varphi_\pi$;

(b) For $x \notin |\mathbb{P}^1(k)|$, $\varphi_x = 1_{G(\mathcal{O}_x)}$;

(c) $\varphi_\infty = 1_{I_{\infty,+}}$;

(d) $\varphi_0 = \chi_k : I_0/I_{0,+} \to \mathbb{C}^\times$

The support conditions imply $P_\varphi(1) = 1$; then set $\Pi = \langle G(F) \cdot P_\varphi \rangle$.

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In general

The assumption that π is compactly induced from $G(\mathcal{O}_F)$ allows us to choose the local groups at 0 and ∞ very simply. In general one shows they can be chosen to guarantee $P_{\varphi}(1) = \varphi(1) = 1$ by an argument on the Bruhat-Tits building of G.

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The case of reductive G reduces easily to the semisimple case. We have not looked seriously at non-split G.

Wild ramification

Theorem

Suppose π is a pure supercuspidal compactly induced from an open compact subgroup that is **small** *in an appropriate sense. Then $\mathcal{L}(\pi)$ is wildly ramified.*

Arguing as before, one gets a non-vanishing Poincaré series that is unramified outside ∞, thus a local system on $\mathbb{A}^1$. By the previous theorem, $\mathcal{L}(\pi)$ is ramified, and since there are no tamely ramified local systems on $\mathbb{A}^1$, the ramification must be wild.

There are more general "small" open compacts – any pro-p open compact is "small" – but the general argument is more subtle.

Mixed supercuspidals

Start with any supercuspidal π and globalize to an automorphic Π such that $\mathcal{L}^{ss}(\Pi)$ is *irreducible* under the adjoint representation. For example, find a $k(Y)$ with points x, y and Π with $\Pi_x = \pi$ and $\Pi_y = Kl$ is the Kloosterman representation considered in Heinlöth-Ngô-Yun, for which the image of $\mathcal{L}(Kl)$ is known to be irreducible.
An observation of Sawin-Templier generalizes purity of the MWF to $\hat{G}$-parameters: it implies that the Weil-Deligne parameter of the restriction to x of $\mathcal{L}^{ss}(\Pi)$ satisfies purity of MWF. In particular, the semisimple parameter $\mathcal{L}(\pi)$ has a completion $(\mathcal{L}(\pi), N)$ that satisfies purity of MWF, as required.

A theorem of Tony Feng

Let K'/K be a cyclic extension of function fields of prime degree ℓ. We suppose ℓ is odd and is a *good prime* for the reductive group G. Explicitly, this means that $\ell > 3$ if G^{ad} is of type G_2, F_4, E_6, or E_7, and $\ell > 5$ if G^{ad} is of type E_8; of course $\ell \neq p$. Let S be a finite set of places of K, and let $\Gamma_{K,S}$ be the Galois group of the maximal extension of K unramified outside S. Using Smith Theory, Feng proves:

Theorem (Feng)

Let Let $\bar{\sigma} : \Gamma_{K,S} \to \hat{G}(\overline{\mathbb{F}}_\ell)$ be an automorphic Langlands parameter (i.e, in the image of Vincent Lafforgue's parametrization mod ℓ). Then the restriction $\bar{\sigma}'$ of $\bar{\sigma}$ to $\Gamma_{K',S}$ is automorphic for $G_{K'}$.

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Application to automorphic base change

Theorem (BFHKT)

Let Π be a cuspidal automorphic representation of $G(\mathbf{A}_K)$. Suppose the Mumford-Tate group of Π is $\hat{G}$. Then there is an integer $b(\Pi)$, depending only on Π, such that, for any good odd prime ℓ for G such that $\ell > b(\Pi)$, and any cyclic extension K'/K of degree ℓ, there is an automorphic representation Π' of $G(\mathbf{A}_{K'})$ such that, at all unramified places v, the representation Π'_v of $G(K'_v)$ is the base change of Π_v.

The bound $\ell > b(\Pi)$ guarantees formal smoothness of the local deformation problems, which allows us to apply the automorphic lifting theorem of [BHKT] (in the minimal case).

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Local result

Definition

Let π be an irreducible admissible representation of $G(F)$ over $\mathbb{Q}_\ell$. For a separable field extension F'/F, we say that irreducible admissible representation Π of $G(F')$ over $\mathbb{Q}_\ell$ is a *base change of* π *to* $G(F')$ if $\sigma_{\pi'} \cong \sigma_\pi|_{W_{F'}}$.

Theorem (BFHKT)

Assume that p is good for G if G is not simply laced. Let π be an irreducible admissible representation of $G(F)$ over $\mathbb{Q}_\ell$. There exists a constant $c(\pi)$ such that for all primes $\ell > c(\pi)$, for any $\mathbb{Z}/\ell\mathbb{Z}$-extension F'/F there exists a base change of π to $G(F')$.

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Deduction from the global theorem

This essentially follows from a new globalization theorem:

Proposition

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Toral representations

For one class of local supercuspidal π the base change can be computed. Chan and Oi have shown that the *toral supercuspidal representations*, a subclass of the *regular supercuspidals* of Kaletha, are compactly induced from generalized Deligne-Lusztig representations.

In particular, they can be realized in the cohomology of certain varieties, and the methods of Smith theory apply to this cohomology. The result is that (under appropriate hypotheses) the base change of such a toral supercuspidal is also toral supercuspidal, and is the one predicted by Kaletha's parametrization.

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Comparison with the trace formula

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Of course these theorems should follow from the stable twisted trace formula over function fields, which I am assuming in other parts of this talk. But for now there is no such formula, and the proof is based on a quite different principle.

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Theorem (H-Ciubotaru)

Let K be a function field over a finite field of k characteristic p, and let G be a connected simple group over K. Let π be a cuspidal automorphic representation of G. Suppose

(1) The local component π_v is generic and unramified at a place v of K such that $G(K_v)$ is an unramified quasi-split group (other than the quasi-split form of a unitary group in $2n+1$ variables);

(2) There is a place u of K such that π_u is tempered;

(3) If G is of type B_n or C_n, we assume $p > 2$. If G is of type F_4 or G_2, we assume $p > 3$.

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Picture of the proof

4.8. F_4. In coordinates in $\mathbb{R}^4$:

$$\begin{array}{lll} \alpha_1 = \frac{1}{2}(1,-1,-1,-1) & \alpha_1^\vee = (1,-1,-1,-1) & \omega_1 = (1,0,0,0) \\ \alpha_2 = (0,0,0,1) & \alpha_2^\vee = (0,0,0,2) & \omega_2 = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) \\ \alpha_3 = (0,0,1,-1) & \alpha_3^\vee = (0,0,1,-1) & \omega_3 = (2,1,1,0) \\ \alpha_4 = (0,1,-1,0) & \alpha_4^\vee = (0,1,-1,0) & \omega_4 = (1,1,0,0). \end{array} \tag{4.20}$$

From [Ci1, Table, p. 126], we see that $r_0 = 9$ and there is only one coroot $\beta^\vee = (1,1,1,-1) = \alpha_1^\vee + 2\alpha_2^\vee + 4\alpha_3^\vee + 2\alpha_4^\vee$ of level 9.

Write $\nu = \sum_{i=1}^4 \nu_i\omega_i \in C_0^{\frac{1}{2}}$, $\nu_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ for all i. Using $\langle\beta^\vee, \nu\rangle < 1$, we get

$$\nu_1 + 2\nu_2 + 4\nu_3 + 2\nu_4 < 1. \tag{4.21}$$

Immediately, $\nu_2 = \nu_3 = \nu_4 = 0$ and $\nu_1 < 1$. So either $\nu_1 = \frac{1}{2}$ or $\nu_1 = 0$. In the first case, we get the point $\nu = \frac{1}{2}\omega_1$, but this lies on the hyperplane $\gamma^\vee = 1$, where $\gamma^\vee = (2,0,0,0)$ is the longest coroot. Thus we are left with the origin.

Idea of the proof

Choose an irreducible representation τ of the L-group ${}^LG$ (we assume G split for simplicity), so $\tau \circ \mathcal{L}(\pi)$ is an ℓ-adic local system $\mathcal{L}(\pi, \tau)$.

In most cases we take τ to be the adjoint representation.

By Deligne's Weil II

$$\mathcal{L}(\pi, \tau) = \oplus_{\chi/\sim} \mathcal{L}(\pi, \tau)_\chi \otimes \chi$$

where each $\mathcal{L}(\pi, \tau)_\chi$ is a sum of punctually pure local systems with integral Frobenius weights and χ is a "fractional" character of $Gal(\bar{k}/k)$.

Suppose for simplicity the tempered π_u is discrete series; then it follows from [GHS] that only the trivial χ contributes.

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