Tuesday 29th November:

- Bernstein decomposition and types

- Cuspidal types

- Covers

§4 Bernstein decomposition $C = \mathbb{C}$

$G = \mathbf{G}(F)$

$\mathcal{B}(G)$ the set of equivalence classes of pairs (M, ρ), with M a Levi subgroup and $\rho \in \mathrm{Irr}_{\mathbb{C}}(M)$ supercuspidal, where

$(M, \rho) \sim (M', \rho') \Longleftrightarrow \exists\, \chi : M \to \mathbb{C}^\times$ unramified s.t. $(M, \rho \otimes \chi)$ is conjugate to (M', ρ') in G

Last time:

Theorem (Harish-Chandra) If $\pi \in \mathrm{Irr}_{\mathbb{C}}(G)$ then $\exists\, (M, \rho)$, with M a Levi subgroup and $\rho \in \mathrm{Irr}_{\mathbb{C}}(M)$ supercuspidal, s.t. $\pi \leq \mathrm{Ind}_{M,P}^{G} \rho$.

The inertial equivalence class $[M, \rho]$ is called the inertial supercuspidal support of π.

Theorem (Bernstein) There is a block decomposition

$$\mathrm{Rep}_C(G) = \prod_{\mathfrak{s}\in\mathcal{B}(G)} \mathrm{Rep}_C^{\mathfrak{s}}(G)$$

where $\mathrm{Rep}_C^{\mathfrak{s}}(G)$ is the full subcategory of representations all of whose irreducible subquotients have inertial supercuspidal support $\mathfrak{s}$.

Example $\mathrm{Rep}_C^{[T,1_T]}(GL_n(F)) = \mathrm{Rep}_C^{I}(GL_n(F))$, where $I = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix}$.

$$\simeq \underbrace{\mathrm{Mod}\text{-}\mathcal{H}_C(G,I)}_{\text{Affine Hecke algebra of type } \tilde{A}} \qquad \text{Iwahori}$$

Remark When char. $C = \ell \neq p$, we have the same decomposition for $GL_n(F)$ (Vignéras) and inner forms (Sécherre-S.).

$$\mathrm{Rep}_{\mathbb{C}}(G) = \prod_{\mathfrak{s}} \mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}}(G)$$

Goal For each $\mathfrak{s}$, find an algebra $\mathcal{H}_{\mathbb{C}}^{\mathfrak{s}}$ such that

$$\mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}}(G) \simeq \mathrm{Mod}\text{-}\mathcal{H}_{\mathbb{C}}^{\mathfrak{s}}$$

By general nonsense, if $\Pi^{\mathfrak{s}}$ is a progenerator for $\mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}}(G)$ then

$$\mathcal{H}_{\mathbb{C}}^{\mathfrak{s}} = \mathrm{End}_G(\Pi^{\mathfrak{s}})$$ will work.

One method (Bernstein, Heiermann, Solleveld)
classical groups — more general

If $\mathfrak{s} = [M, \rho]$ and $\rho_0 \hookrightarrow \mathrm{Res}^M_{M_0} \rho$ is irreducible

(where M_0 is the subgroup generated by all compact subgroups)

then $\Pi^{\mathfrak{s}} = \mathrm{Ind}^G_{M,P}(\mathrm{ind}^M_{M_0} \rho_0)$ will do, and one can then describe $\mathrm{End}_G(\Pi^{\mathfrak{s}})$.

Type theory looks for another way.

Type theory Find (J,λ) with J compact open and $\lambda \in \mathrm{Irr}_{\mathbb{C}}(J)$ such that

$$\pi \in \mathrm{Irr}_{\mathbb{C}}^{\mathfrak{s}}(G) \iff \mathrm{Hom}_J(\lambda,\pi) \neq 0$$

Call such (J,λ) an $\mathfrak{s}$-type.

Then
$$\begin{cases} \Pi^{\mathfrak{s}} = \mathrm{ind}_J^G \lambda & \text{is a progenerator} \\ \mathcal{H}_{\mathbb{C}}^{\mathfrak{s}} = \mathrm{End}_G(\mathrm{ind}_J^G \lambda) =: \mathcal{H}(G,\lambda) & \lambda\text{-spherical Hecke algebra} \end{cases}$$

Advantages:

- The type can give precise information about $\pi \in \mathrm{Irr}_{\mathbb{C}}^{\mathfrak{s}}(G)$.
- The same methods can be used for C-representations when $\ell = \mathrm{char}\, C \neq 0$

  Note: $\mathrm{ind}_J^G \lambda$ may no longer be a progenerator but still get a bijection of simple objects.

§5 Cuspidal Types

Suppose $\begin{cases} \tilde{J} \text{ is a compact-mod-centre subgroup of } G \\ \tilde{\lambda} \in \mathrm{Irr}_c(\tilde{J}) \end{cases}$

Theorem (Carayol) If $\mathrm{ind}_{\tilde{J}}^{G} \tilde{\lambda}$ is irreducible then it is cuspidal.

[Reason: $(\pi, V) \in \mathrm{Irr}_c(G)$ is cuspidal iff it is Z-compact,

i.e. $\forall v \in V$, $\tilde{v} \in \tilde{V}$ the map $G \longrightarrow \mathbb{C}$, $g \longmapsto \langle \pi(g)v, \tilde{v} \rangle$ has compact-mod-centre support]

($\tilde{V}$: smooth dual)

$\tilde{J}$ compact-mod-centre

$\tilde{\lambda} \in \mathrm{Irr}_{\mathbb{C}}(\tilde{J})$

Theorem If $\mathrm{ind}_{\tilde{J}}^{G} \tilde{\lambda}$ is irreducible then it is cuspidal.

By the Mackey formula

$$\mathrm{End}_G(\mathrm{ind}_{\tilde{J}}^{G} \tilde{\lambda}) \simeq \bigoplus_{\tilde{J}\backslash G/\tilde{J}} \mathrm{Hom}_{\tilde{J} \cap {}^g\tilde{J}}(\tilde{\lambda}, {}^g\tilde{\lambda}) \supseteq \mathrm{End}_{\tilde{J}}(\tilde{\lambda})$$

so (by Schur's lemma) a necessary condition for irreducibility is

$$\underbrace{I_G(\tilde{\lambda})}_{\text{intertwining of } \tilde{\lambda}} := \{ g \in G : \mathrm{Hom}_{\tilde{J} \cap {}^g\tilde{J}}(\tilde{\lambda}, {}^g\tilde{\lambda}) \neq 0 \} = \tilde{J}$$

- Controlling the intertwining is crucial to constructing cuspidal rep$^{\underline{n}}$s.

$\tilde{J}$ compact-mod-centre subgroup of G

$\tilde{\lambda} \in \mathrm{Irr}_c(\tilde{J})$

Theorem If $\mathrm{ind}_{\tilde{J}}^{G}\tilde{\lambda}$ is irreducible then it is cuspidal.

Theorem (Bushnell-Kutzko) If $\pi = \mathrm{ind}_{\tilde{J}}^{G}\tilde{\lambda}$ is irreducible cuspidal and

$$\begin{cases} J = \tilde{J}_0 & \text{maximal compact subgroup} \\ \lambda \hookrightarrow \tilde{\lambda}|_J & \text{irreducible} \end{cases}$$

then (J,λ) is a cuspidal type for $[G,\pi]$.

(i.e. $\mathrm{Hom}_J(\lambda,\pi') \neq 0 \Leftrightarrow \pi' \simeq \pi \otimes \chi$ with χ unramified.)

Examples $G = GL_n(F)$

(1) **Depth zero** $J = GL_n(\mathcal{O}) \subset \tilde{J} = ZJ = N_G(J)$

$$J^{0+} \hookrightarrow J \twoheadrightarrow GL_n(k_F)$$

If • λ is the inflation to J of an irreducible cuspidal rep$^{\underline{n}}$ of $GL_n(k_F)$

• $\tilde{\lambda}$ is any extension to $\tilde{J}$

then $\pi = \mathrm{ind}_{\tilde{J}}^{G} \tilde{\lambda}$ is irreducible cuspidal.

The same idea works for arbitrary G (Moy-Prasad, Morris).

[J^0 a maximal parahoric, $\tilde{J} = N_G(J^0)$

λ^0 the inflation of an irred. cuspidal of the reductive quotient, extended to ZJ

$\mathrm{ind}_{ZJ}^{\tilde{J}} \lambda^0 \twoheadrightarrow \tilde{\lambda}$ irreducible]

(2) Positive depth $\quad G = GL_4(F)$

$$\beta = \left(\begin{array}{cc|cc} & & & \varpi^{-2} \\ & & \varpi^{-2} & \\ \hline & \varpi^{-1} & & \\ \varpi^{-1} & & & \end{array}\right) \in \mathrm{Lie}(G) \qquad \text{so} \quad \beta^2 = \varpi^{-3}$$

$E = F[\beta]$, ramified quadratic $/F$ with $\varpi_E = \varpi^2\beta$

$$G' = \mathrm{Cent}_G(E) \xrightarrow{\sim} GL_2(E)$$

$$K = \left(\begin{array}{c|c} \mathfrak{o} & \mathfrak{o} \\ \hline \mathfrak{p} & \mathfrak{o} \end{array}\right)^\times \qquad \text{has} \quad K \cap G' \simeq GL_2(\mathfrak{o}_E)$$

$$\triangledown$$

$$K^1 = 1 + \left(\begin{array}{c|c} \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p}^2 & \mathfrak{p} \end{array}\right)$$

We use β to define a character ψ_β of K^1:

$$\psi_\beta(1+X) = \psi \circ \mathrm{tr}(\beta X) \quad \text{, intertwined by } G'.$$

$G = GL_4(F)$, $E = F[\beta]$ with $\beta^2 = \varpi^{-3}$, $G' = \mathrm{Cent}_G(E) \simeq GL_2(E)$

K such that $K \cap G' \simeq GL_2(\mathcal{O}_E)$

$$\psi_\beta(1+X) = \psi \circ \mathrm{tr}(\beta X) \qquad \text{for } 1+X \in K^1$$

Extend ψ_β to a character θ of $J^{0+} = (K^{0+} \cap G')K^1$ such that $\theta|_{K^{0+} \cap G'}$ factors through $\det_{GL_2(E)}$.

[i.e. it is intertwined by all of $GL_2(E)$.]

Extend (carefully) further to κ of $J = (K \cap G')K^1$.

$$J^{0+} \hookrightarrow J \twoheadrightarrow GL_2(k_E)$$

Take ρ the inflation to J of an irreducible cuspidal repⁿ of $GL_2(k_E)$

$\lambda = \kappa \otimes \rho$

Then λ extends to $\tilde{\lambda}$ of $\tilde{J} = E^\times J$ and $\mathrm{ind}_{\tilde{J}}^{G} \tilde{\lambda}$ is irreducible cuspidal.

General idea G' a twisted Levi subgroup such that $Z(G')/Z(G)$ is compact!!

Some "arithmetic data" gives κ

$\lambda = \kappa \otimes \rho$ for ρ an irreducible cuspidal representation of some finite reductive group

Theorem There is an exhaustive list of cuspidal types (J,λ) for:

- $GL_n(F)$ — Bushnell–Kutzko, Vignéras
- $SL_n(F)$ — Bushnell–Kutzko, Cui
- $GL_m(D)$ — Sécherre–S., Mínguez–Sécherre
- "Classical groups" with $p \neq 2$ — Kurinczuk–Skodlerack–S.
- G split over a tame extension, $p \nmid |W_G|$ (Weyl group) — Yu, Kim, Fintzen

In all cases, if (J,λ) and (J',λ') are cuspidal types for the same cuspidal representation then they are conjugate in G.

Theorem (Henniart–Vignéras)

- Do not need C to be algebraically closed
- (J,λ) is a supercuspidal type iff ρ is supercuspidal.

§6 Covers $C = \mathbb{C}$

(M, ρ) a supercuspidal pair gives us

$\mathfrak{s}$ inertial equivalence class in $\mathfrak{B}(G)$

$\mathfrak{s}_M$ inertial equivalence class in $\mathfrak{B}(M)$

Suppose (J_M, λ_M) is a cuspidal type for $\mathfrak{s}_M$.

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}}(G) \arrow[r, dashed, "\sim"] & \mathrm{Mod}\text{-}\mathcal{H}(G, \lambda) \\
\mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}_M}(M) \arrow[u, "\mathrm{Ind}_{M,P}^{G}"] \arrow[r, "\sim"] & \mathrm{Mod}\text{-}\mathcal{H}(M, \lambda_M) \arrow[u, dotted]
\end{tikzcd}$$

How do we find (J, λ) ?

Definition A pair (J, λ) is called a cover of (J_M, λ_M) if:

(1) for $P = MN$ with opposite $\bar{P} = M\bar{N}$, there is an Iwahori decomposition $J = (J \cap \bar{N})(J \cap M)(J \cap N)$ with $J \cap M = J_M$.

(2) $\lambda|_{J_M} = \lambda_M$ and λ is trivial on $J \cap \bar{N}$, $J \cap N$.

(3) Technical condition: there is invertible $\phi \in \mathcal{H}(G, \lambda)$ such that $\operatorname{supp}\phi = J \zeta J$ with $\zeta \in Z(M)$ strongly positive.

$$\left[\ \bigcap_{r \geq 0} \zeta^{r} (J \cap N) \zeta^{-r} = \{1\} = \bigcap_{r \geq 0} \zeta^{-r} (J \cap \bar{N}) \zeta^{r}\ \right]$$

Archetype: $(I, \mathbb{1})$ is a cover of $(T_0, \mathbb{1})$ for $M = T$ in $GL_n(F)$.

$$I = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix} \qquad T_0 = \begin{pmatrix} \mathcal{O}^\times & & \\ & \ddots & \\ & & \mathcal{O}^\times \end{pmatrix}$$

$$\zeta = \begin{pmatrix} \varpi^{n-1} & & & \\ & \varpi^{n-3} & & \\ & & \ddots & \\ & & & \varpi^{1-n} \end{pmatrix}$$

Theorem (Bushnell–Kutzko) Suppose (J,λ) is a cover of (J_M, λ_M) an $\mathfrak{s}_M$-type. Then

- (J,λ) is an $\mathfrak{s}$-type;
- there is an algebra embedding $t_P : \mathcal{H}(M, \lambda_M) \hookrightarrow \mathcal{H}(G,\lambda)$

such that

$$\begin{tikzcd}
\mathrm{Rep}_c^{\mathfrak{s}}(G) \arrow[r, "\sim"] & \text{Mod-}\mathcal{H}(G,\lambda) & \mathrm{Hom}_{\mathcal{H}(M,\lambda_M)}(\mathcal{H}(G,\lambda), X) \\
\mathrm{Rep}_c^{\mathfrak{s}_M}(M) \arrow[r, "\sim"] \arrow[u, "\mathrm{Ind}_{M,P}^G"] & \text{Mod-}\mathcal{H}(M,\lambda_M) \arrow[u, "(t_P)_*"'] & X \arrow[u, mapsto]
\end{tikzcd}$$

Example $G = SL_2(F)$, $p \neq 2$

$\cup\!\!\!\shortmid$

$M = \left\{ \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} : x \in F^\times \right\} \simeq F^\times$ and $\rho = 1\!\!1_M$

$J_M = \mathcal{O}^\times$, $\lambda_M = 1\!\!1$ so that (J_M, λ_M) is an $\mathfrak{s}_M$-type

$J = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix} \cap G$ Iwahori subgroup, $\lambda = 1\!\!1_J$.

$\cap\!\!\shortmid$

$K = SL_2(\mathcal{O}) = J \cup JsJ$ where $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and

$$\mathcal{H}(G, 1\!\!1_J) \longleftrightarrow \mathcal{H}(K, 1\!\!1_J) = \mathrm{End}_K(\mathrm{ind}_J^K 1\!\!1_J) \simeq \underbrace{\mathrm{End}_{SL_2(k)}\left(\mathrm{ind}_{B(k)}^{SL_2(k)} 1\!\!1\right)}$$

2-dimensional, generated by $\bar{T}$ s.t.

$$T \longleftarrow\!\!\shortmid \bar{T}$$

$\mathrm{supp}(\bar{T}) = B(k)\, \bar{s}\, B(k)$

$(\bar{T} + 1)(\bar{T} - q) = 0$

$J = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi & \mathcal{O}^\times \end{pmatrix} \cap G \hookrightarrow K = SL_2(\mathcal{O})$, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathcal{H}(G, \mathbb{1}_J) \overset{\cdot}{\longleftrightarrow} \mathcal{H}(K, \mathbb{1}_J) \simeq \underbrace{\mathrm{End}_{SL_2(k)}\left(\mathrm{ind}_{B(k)}^{SL_2(k)} \mathbb{1}\right)}$$

$$T \longleftarrow \overline{T} \text{ s.t. } \mathrm{supp}(\overline{T}) = B(k)\, \overline{s}\, B(k)$$
$$(\overline{T}+1)(\overline{T}-q) = 0$$

Also $J \hookrightarrow K' = \begin{pmatrix} \mathcal{O} & \varpi^{-1} \\ \varpi & \mathcal{O} \end{pmatrix} \cap G = \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix} K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$, $s' = \begin{pmatrix} 0 & -\varpi^{-1} \\ \varpi & 0 \end{pmatrix}$

so we also find

$$T' \in \mathcal{H}(G, \mathbb{1}_J) \text{ with } \mathrm{supp}(T') = J s' J$$
$$(T'+1)(T'-q) = 0$$

Then $\Phi = TT'$ is invertible with support

$$J s J s' J = J s (J \cap \overline{N}) \underbrace{s^{-1}} \underbrace{s s'} (s')^{-1} (J \cap B) s' J = J \underbrace{s s'} J$$
$$\zeta = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$$

$J = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix} \cap G \subseteq G = SL_2(F)$, $\quad J_M = \mathcal{O}^\times \subseteq M = \left\{ \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} : x \in F^\times \right\}$

Then

$$\begin{array}{ccl} \mathcal{H}(M, \mathbb{1}_{J_M}) & \cong & \mathbb{C}[Z, Z^{-1}] \\ \big\downarrow & & \quad Z \\ & & \quad \downarrow \\ & & \quad TT' \\ \mathcal{H}(G, \mathbb{1}_J) & = & \langle T, T' : (T+1)(T-q) = (T'+1)(T'-q) = 0 \rangle \end{array}$$

$\mathcal{H}(G, \mathbb{1}_J)$ has 4 characters, $T \mapsto -1$ or q and $T' \mapsto -1$ or q, so these are the irreducible subquotients of reducible inductions coming from

$$Z \longmapsto 1, -q, -q, q^2$$

i.e. for χ unramified

$\operatorname{Ind}_B^G \chi$ is reducible $\iff$ $\underbrace{\chi = \mathbb{1} \text{ or } |\cdot|^2}_{\substack{\text{length 2 indecomposable} \\ \mathbb{1}_G,\, St_G \text{ as components}}}$ or $\underbrace{\chi = \omega_0 |\cdot|}_{\substack{\text{length 2} \\ \text{semisimple}}}$

unramified quadratic char.

Theorem There are constructions of covers in all the same situations as for cuspidal types.

(except maybe quaternionic classical groups)

- $GL_n(F)$ — Bushnell–Kutzko
- $SL_n(F)$ — Goldberg–Roche
- $GL_m(D)$ — Sécherre–S.
- Classical groups, $p \neq 2$ — Miyauchi–S.
- G split over a tame extension, $p \nmid |W_G|$ — Kim–Yu, Fintzen

However computing their Hecke algebras seems hard in general.