Wednesday 30th November:

- Local Langlands for GL_n
- Local Langlands more generally
- Endo-classes
- Other questions

§7 Local Langlands for GL_n

F non-archimedean local field

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] & \mathcal{G}_F \arrow[r] & \mathrm{Gal}(\overline{k}_F/k_F) \simeq \hat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_F \arrow[r] \arrow[u, equal] & W_F \arrow[r] \arrow[u, hook] & \langle \mathrm{Frob} \rangle \arrow[r] \arrow[u, hook] & 1 \\
 & \text{open} & \text{locally compact} & &
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{F} \arrow[d, dash] \arrow[d, bend left, dash, "\mathcal{P}_F"] \arrow[dd, bend right, dash, "I_F"'] \arrow[ddd, bend right=60, dash, "\mathcal{G}_F"'] \\
F^{tr} \arrow[d, dash] \\
F^{ur} \arrow[d, dash] \\
F
\end{tikzcd}$$

Local Class Field Theory

There is a natural isomorphism of topological groups $F^\times \xrightarrow{\sim} W_F^{ab}$

which gives a natural bijection

$$\{\text{continuous characters } F^\times \to \mathbb{C}^\times\} \longleftrightarrow \{\text{continuous characters } W_F \to \mathbb{C}^\times\}$$

LCFT: $\left\{\begin{array}{l}\text{smooth irreducible complex}\\ \text{representations of } GL_1(F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{1-dimensional continuous}\\ \text{complex representations of } W_F\end{array}\right\}$

LLC:

$$\begin{tikzcd} \mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \arrow[r] & \Phi^{ss}(GL_n(F)) = \left\{\begin{array}{l}\varphi : W_F \to GL_n(\mathbb{C}) \text{ continuous}\\ \text{semisimple}\end{array}\right\}/\sim \\ \mathrm{Cusp}_{\mathbb{C}}(GL_n(F)) \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & \Phi^{cusp}(GL_n(F)) = \{\text{irreducible } \varphi\}/\sim \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\left.\begin{array}{l}\pi_\varphi = \pi_1 \otimes \cdots \otimes \pi_r \text{ cuspidal}\\ \text{irreducible of } M = \prod_{i=1}^{r} GL_{n_i}(F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\varphi : W_F \to GL_n(\mathbb{C}) \text{ semisimple}\\ \varphi = \bigoplus_{i=1}^{r} \varphi_i \quad \text{with } \varphi_i : W_F \to GL_{n_i}(\mathbb{C}) \text{ irreducible}\end{array}\right.$

But $\underline{\mathrm{I}}\mathrm{nd}_{M,P}^{G}\, \pi_\varphi$ is not irreducible in general

LLC for GL_n (Laumon-Rapoport-Stuhler, Harris-Taylor, Henniart)

$$\mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \xrightarrow{\sim} \Phi(GL_n(F)) = \left\{ \begin{array}{l} \varphi : W_F \times SL_2(\mathbb{C}) \to GL_n(\mathbb{C}) \\ \varphi|_{W_F} \text{ semisimple}, \ \varphi|_{SL_2(\mathbb{C})} \text{ algebraic} \end{array} \right\} / \sim$$

$$\pi \longmapsto \varphi_\pi$$

"natural" bijections — satisfying a long list of properties:

- Central character: $\omega_\pi \longleftrightarrow \det(\varphi_\pi)$ under LCFT
- Twisting $\varphi_{\pi \otimes \chi} = \varphi_\pi \otimes \chi$ for χ character of $\left\{ \begin{array}{l} F^\times \\ W_F \end{array} \right.$
- Contragredient $\varphi_{\pi^\vee} = \varphi_\pi^\vee$
- L & ε-factors $L(\pi \times \pi', s) = L(\varphi_\pi \otimes \varphi_{\pi'}, s)$

$$\varepsilon(\pi \times \pi', s, \psi) = \varepsilon(\varphi_\pi \otimes \varphi_{\pi'}, s, \psi)$$

LLC for GL_n

$$\begin{tikzcd}
\mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \arrow[r, leftrightarrow, "\sim"] & \Phi(GL_n(F)) \\
\cup & \cup \\
\mathrm{DS}_{\mathbb{C}}(GL_n(F)) \arrow[r, leftrightarrow, "\sim"] & \Phi^{ds}(GL_n(F)) = \{\text{indecomposable } \varphi = \varphi_0 \otimes \mathrm{St}_a\}/\cong \\
\cup & \cup \\
\mathrm{Cusp}_{\mathbb{C}}(GL_n(F)) \arrow[r, leftrightarrow, "\sim"] & \Phi^{cusp}(GL_n(F)) = \{\text{irreducible } \varphi : W_F \to GL_n(\mathbb{C})\}/\cong
\end{tikzcd}$$

$$\pi \longmapsto \varphi_\pi$$

It is compatible with parabolic induction:

If $\pi \hookrightarrow \underbrace{\iota^G_{M,P}}_{\text{normalized parabolic induction}}(\pi_1 \otimes \cdots \otimes \pi_r)$ then $\underbrace{\varphi_\pi|_{W_F}} = \varphi_{\pi_1} \oplus \cdots \oplus \varphi_{\pi_r}$

where $W_F \hookrightarrow W_F \times SL_2(\mathbb{C})$

$$w \longmapsto \left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right)$$

LLC for GL_n : depth

$$\begin{tikzcd}
\mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \arrow[r, leftrightarrow] & \Phi(GL_n(F)) \\
\pi \arrow[r, mapsto] & \varphi_\pi \\
\rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \\
\underbrace{\mathrm{Irr}^0_{\mathbb{C}}(GL_n(F))} \arrow[r, leftrightarrow] & \Phi^{\mathrm{tame}}(GL_n(F)) = \{\varphi \text{ s.t. } \varphi|_{P_F} \text{ is trivial}\}
\end{tikzcd}$$

depth 0 representations (π, V) : i.e. $V^{K^{0+}} \neq 0$, where $K^{0+} = 1 + \mathrm{Mat}_n(\mathfrak{p})$

More generally the depth is preserved by LLC for GL_n :

$$\ell(\pi) = \inf\left\{ \begin{array}{l} r \geq 0 : V^{\mathcal{P}^{r+}} \neq 0 \text{ for} \\ \text{some parahoric } \mathcal{P} \end{array} \right\}$$

$$\ell(\varphi) = \inf\{ r \geq 0 : \varphi|_{\underbrace{W_F^{r+}}} \text{ is trivial} \}$$

filtration

with $W_F^0 = I_F$, $W_F^{0+} = P_F$

Restriction to inertia

$$\mathrm{Cusp}_{\mathbb{C}}(GL_n(F)) \longleftrightarrow \Phi^{\mathrm{cusp}}(GL_n(F))$$

$$\pi \longmapsto \varphi_\pi$$

Then

- $\varphi_\pi|_{I_F} = \varphi_{\pi'}|_{I_F} \iff \pi \simeq \pi' \otimes \chi$, χ unramified
- **(Paškūnas)** Let $\Psi : I_F \longrightarrow GL_n(\mathbb{C})$ be the restriction of some Langlands parameter.
  There is a unique irreducible representation λ_Ψ of $GL_n(\mathcal{O}_F) = K$ such that

$$\mathrm{Hom}_K(\lambda_\Psi, \pi) \neq 0 \iff \varphi_\pi|_{I_F} = \Psi$$

Restriction to wild inertia (Bushnell–Henniart)

If $\varphi : W_F \to GL_n(\mathbb{C})$ is irreducible then $\varphi|_{\mathcal{P}_F}$ is a multiple of a W_F-orbit in $\mathrm{Irr}_{\mathbb{C}}(\mathcal{P}_F)$.

$$\begin{tikzcd}
\mathrm{Cusp}_{\mathbb{C}}(GL_n(F)) \arrow[r, "\sim"] \arrow[d] & \Phi^{\mathrm{cusp}}(GL_n(F)) \arrow[d, "\mathrm{Res}"] \\
\text{``Endoclasses''} \arrow[r, leftrightarrow, "\sim"] & W_F \backslash \mathrm{Irr}_{\mathbb{C}}(\mathcal{P}_F)
\end{tikzcd}$$

or

$$\begin{tikzcd}
\mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \arrow[r, "\sim"] \arrow[d] & \Phi(GL_n(F)) \arrow[d, "\mathrm{Res}"] \\
\text{``Endoparameters''} \arrow[r, leftrightarrow, "\sim"] & \{\psi : \mathcal{P}_F \to GL_n(\mathbb{C})\}^{W_F}
\end{tikzcd}$$

Explicit LLC for GL_n (Bushnell – Henniart)

$\varphi \in \Phi^{cusp}(GL_n(F))$ is **essentially tame** if $\varphi|_{P_F}$ is a sum of characters

$(\pi, V) \in \mathrm{Cusp}(GL_n(F))$ is **essentially tame** if the data involved in its construction is tame

$$\left[\iff p \nmid \frac{n}{t(\pi)} \quad \text{where } t(\pi) = \#\{\text{unramified } \chi : \pi \otimes \chi\circ\det \simeq \pi\} \right]$$

Bushnell – Henniart make the correspondence explicit for essentially tame representations/parameters

§ 8 Local Langlands more generally

$G = G(F)$ has a complex dual group $\hat{G} = \hat{G}(\mathbb{C})$

[Take based root datum for G and swap roots and coroots.]

G	$GL_n(F)$	$SL_n(F)$	$Sp_{2n}(F)$	$SO_{2n}(F)$	…
$\hat{G}$	$GL_n(\mathbb{C})$	$PGL_n(\mathbb{C})$	$SO_{2n+1}(\mathbb{C})$	$SO_{2n}(\mathbb{C})$	

L-group ${}^LG = \hat{G} \rtimes W_F$ (direct product for split groups)

Definition A parameter for ${}^LG$ is continuous $\varphi : W_F \times SL_2(\mathbb{C}) \to {}^LG$ s.t.

- $\varphi(w) \in \hat{G}\, w \quad \forall w \in W_F$
- $\varphi(w)$ is semisimple $\forall w \in W_F$
- $\varphi|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to \hat{G}$ is a homomorphism of complex alg. groups

$\Phi(G) = \{$ parameters for ${}^LG$ which are relevant for $G \} / \hat{G}$-conjugacy

LLC predicts

$$\mathrm{Irr}_e(G) \twoheadrightarrow \Phi(G)$$

with finite fibres (L-packets), satisfying various naturality conditions.

Warning L-packets can contain cuspidal and non-cuspidal representations
(look at discrete series representations instead)

To parametrize representations in an L-packet, we need more:

Enhanced parameters (ϕ, ρ), where ρ is an irreducible
representation of some finite group S_ϕ

Arthur There is such an LLC for symplectic and special orthogonal groups

Semisimple LLC

Fargues–Scholze give a map

$$\mathrm{Irr}_{\mathbb{C}}(G) \longrightarrow \Phi^{ss}(G) = \{\varphi : W_F \longrightarrow {}^LG\}/\sim$$

$$\pi \mapsto \varphi_\pi^{ss}$$

with the expected properties, for arbitrary G.

Expect that

$$\varphi_\pi^{ss}(w) = \varphi_\pi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right)$$

Hecke algebras (Aubert, Moussaoui, Solleveld)

$G = \mathcal{G}(F)$ symplectic, (special) orthogonal, general (s)pin, unitary

$\mathfrak{s} \in \mathcal{B}(G) \rightsquigarrow$ block $\mathrm{Rep}_{\mathfrak{c}}^{\mathfrak{s}}(G) \simeq \text{Mod-}\mathcal{H}_{\mathfrak{c}}^{\mathfrak{s}}$

Also (for arbitrary G) define cuspidal support of an enhanced Langlands parameter, and hence get Bernstein components $\Phi_e(G)^{\mathfrak{s}^\vee}$ and, for each such, build a Hecke algebra $\mathcal{H}_{\mathfrak{c}}^{\mathfrak{s}^\vee}(z)$ (z an indeterminate) and

$$\Phi_e(G)^{\mathfrak{s}^\vee} \xleftrightarrow{\sim} \mathrm{Irr}\left(\mathcal{H}_{\mathfrak{c}}^{\mathfrak{s}^\vee}(q^{1/2})\right)$$

Match $\mathrm{Rep}_{\mathfrak{c}}^{\mathfrak{s}}(G)$ to $\Phi_e(G)^{\mathfrak{s}^\vee}$ using Arthur-Moeglin; then

$$\left(\mathcal{H}_{\mathfrak{c}}^{\mathfrak{s}}\right)^{op} \simeq \mathcal{H}_{\mathfrak{c}}^{\mathfrak{s}^\vee}(q^{1/2})$$

Use this to build LLC

$$\mathrm{Irr}(G) \longrightarrow \Phi_e(G)$$

using only Arthur-Moeglin for cuspidals

§9 Endoclasses

The construction of cuspidal representations of $G = \underset{\mathrm{Aut}_F(V)}{\underset{\wr\wr}{GL_n(F)}}$ involves simple characters θ.

Data: $E = F[\beta]$ extension of degree d

$\varphi : E \hookrightarrow \mathrm{End}_F(V)$ embedding

x point in the Bruhat-Tits building $\widetilde{\underline{I}}(G_\varphi, E) \hookrightarrow \widetilde{\underline{I}}(G, F)$

from which Bushnell-Kutzko defined

$\mathcal{C}(x, \varphi|\beta)$ a set of simple characters of a pro-p group H^{0+}

Remarkable Fact 1 There are canonical bijections $\mathcal{C}(x, \varphi|\beta) \xrightarrow[\text{transfer}]{\sim} \mathcal{C}(x', \varphi'|\beta)$

and $\theta \leftrightarrow \theta'$ iff 1_G intertwines θ with θ'.

But θ does not determine $\varphi|\beta$.

Suppose we have simple characters $\theta \in \mathcal{C}(x, \varphi|\beta))$ and $\theta' \in \mathcal{C}(x', \varphi'|\beta'))$.

Definition θ, θ' are endo-equivalent if there exist

$$\left[\begin{tikzcd} \varphi_1 : F[\beta] \arrow[r, hook] & \mathrm{End}_F(V) \\ \varphi_1' : F[\beta'] \arrow[ur] & \end{tikzcd}\right.$$

and x_1, x_1' such

the transfers of θ, θ' to $\mathcal{C}(x_1, \varphi_1(\beta)), \mathcal{C}(x_1', \varphi_1'|\beta'))$ intertwine.

Remarkable Facts 2 (Bushnell–Henniart) Suppose θ, θ' are endo-equivalent.

(i) The transfers of θ, θ' to any common space V intertwine.

(ii) The transfers of θ, θ' to any common (V, x) are conjugate.

Definition An endo class is an endo-equivalence class of simple characters

$\mathcal{E}(F)$ is the set of endo-classes

If $\Theta \in \mathcal{E}(F)$ contains a simple character $\theta \in \mathcal{C}(x, \varphi|\beta))$ then

$$d_\Theta = [F[\beta]:F], \quad e_\Theta = e(F(\beta)/F), \quad f_\Theta = f(F(\beta)/F)$$

depend only on Θ.

Ramification Theorem (Bushnell–Henniart)

$$\begin{tikzcd}
\pi \arrow[d, mapsto] & \bigsqcup_{n\geq 1} \mathrm{Cusp}_{\mathbb{C}}(GL_n(F)) \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \mathrm{Irr}_{\mathbb{C}}(W_F) \arrow[d] \\
{[\theta]} & \mathcal{E}(F) \arrow[r, leftrightarrow, "\sim"] & W_F \backslash \mathrm{Irr}_{\mathbb{C}}(\mathcal{P}_F)
\end{tikzcd}$$

θ simple character in the construction of π

Definition $\mathcal{EP}(GL_n(F)) = \{ (\Theta_i, n_i)_{i\in I} \; : \; \sum n_i d_{\Theta_i} = n \}$ degree n endo-parameters

Then

$$\begin{tikzcd}
\mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \Phi(GL_n(F)) \arrow[d, "\mathrm{Res}_{\mathcal{P}_F}"] \\
\mathcal{EP}(GL_n(F)) \arrow[r, leftrightarrow, "\sim"] & \{\psi : \mathcal{P}_F \to GL_n(\mathbb{C})\}^{W_F}
\end{tikzcd}$$

either: take endoclasses of cuspidal support

or: generalize simple characters to allow β to be semisimple $\Big\}$ then $\mathcal{EP}(GL_n(F))$ is the intertwining classes of semisimple characters

Category decomposition (Vignéras, Helm–Kurinczuk–Skodlerack–S.)

$$\mathrm{Rep}_c(GL_n(F)) = \prod_{\mathfrak{t} \in \mathcal{EP}(GL_n(F))} \mathrm{Rep}_c^{\mathfrak{t}}(GL_n(F))$$

For suitable θ with endo-parameter $\mathfrak{t}$,

$\mathrm{ind}_{H^{0+}}^{GL_n(F)} \theta$ is a progenerator of $\mathrm{Rep}_c^{\mathfrak{t}}(GL_n(F))$

Remarks (1) This refines the depth decomposition $\prod_{r \geq 0} \mathrm{Rep}_c^{r}(GL_n(F))$. (Vignéras)

(2) Only needs C to be a commutative $\mathbb{Z}[\frac{1}{p}]$-algebra.

(3) This generalizes to classical groups G $(p \neq 2)$ to give a set of endoparameters $\mathcal{EP}(G)$. We expect

$$\mathcal{EP}(G) \xleftrightarrow{\sim} \{\text{enhanced wild parameters for } G\}$$

§ 10 Other questions

Explicit LLC:

When can we make LLC explicit, with the data used to build representation?

- Depth zero — DeBacker–Reeder
- Tame — Kaletha

⋮

Explicit transfer: If $\eta : {}^{L}H \longrightarrow {}^{L}GL_n$ then we should get

$$\begin{tikzcd}
\mathrm{Irr}_e(H) \arrow[r, two heads, "LLC"] \arrow[d, dashed, "transfer"] & \Phi(H) \arrow[d, "\eta\circ"] \\
\mathrm{Irr}_e(GL_n(F)) \arrow[r, leftrightarrow, "LLC"] & \Phi(G)
\end{tikzcd}$$

Make this explicit.

Endoparameters for arbitrary G: are there such things which

- reflect restriction to wild inertia
- behave functorially in the group
- give a decomposition of $\mathrm{Rep}_c(G)$

Moreover, hopefully for t an endo-parameter

$$\mathrm{Rep}_c^t(G) \simeq \mathrm{Rep}_c^0(G_t)$$

depth 0 rep's of another group G_t

[Chinello: true for $GL_m(D)$]

Reduction to a unipotent block?

$$\mathrm{Rep}_c^{s}(G) \simeq \mathrm{Rep}_c^{s_u}(G_s)$$

for some group G_s and unipotent inertial cuspidal support s_u.

Other correspondences

Jacquet-Langlands: bijection between essentially square-integrable irreducible representations of $GL_n(F)$ and an inner form.

Explicit description? · Dotto does so at the inertial level (it preserves endo-classes)

Compatibility with mod-ℓ congruences / reduction modulo ℓ: Minguez-Sécherre

Theta correspondence: bijection between certain representations of a dual pair (G, G')

Explicit description? · Loke-Ma in the tame case

Compatibility with mod-ℓ congruences / reduction modulo ℓ?