

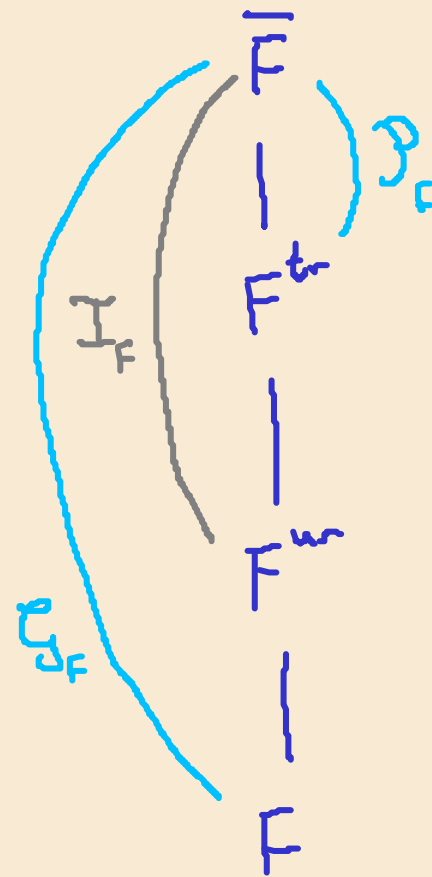
Wednesday 30<sup>th</sup> November:

- Local Langlands for  $GL_n$
- Local Langlands more generally
- Endo-classes
- Other questions

# §7 Local Langlands for GL<sub>n</sub>

F non-arch: median local field

$$\begin{array}{ccccccc}
 1 & \longrightarrow & I_F & \longrightarrow & \mathcal{O}_F & \longrightarrow & \text{Gal}(\overline{k}_F/k_F) \cong \hat{\mathbb{Z}} \longrightarrow 1 \\
 & & \parallel & & \updownarrow & & \downarrow \\
 1 & \longrightarrow & I_F & \longrightarrow & W_F & \longrightarrow & \langle \text{Frob} \rangle \longrightarrow 1 \\
 & & \text{open} & & \text{locally compact} & & 
 \end{array}$$



## Local Class Field Theory

of topological groups

There is a natural isomorphism

$$F^{\times} \xrightarrow{\sim} W_F^{\text{ab}}$$

which gives a natural bijection

$$\{\text{continuous characters } F^{\times} \rightarrow \mathbb{C}^{\times}\} \longleftrightarrow \{\text{continuous characters } W_F \rightarrow \mathbb{C}^{\times}\}$$

LCFT:  $\left\{ \begin{array}{l} \text{smooth irreducible complex} \\ \text{representations of } \mathrm{GL}_n(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{1-dimensional continuous} \\ \text{complex representations of } \mathcal{W}_F \end{array} \right\}$

LLC:  $\mathrm{Irr}_c(\mathrm{GL}_n(F)) \longrightarrow \underline{\Phi}^{\mathrm{ss}}(\mathrm{GL}_n(F)) = \left\{ \begin{array}{l} \varphi: \mathcal{W}_F \rightarrow \mathrm{GL}_n(\mathbb{C}) \text{ continuous} \\ \text{semisimple} \end{array} \right\} / \sim$

$\bigcup$

$\mathrm{Cusp}_c(\mathrm{GL}_n(F)) \longleftrightarrow \underline{\Phi}^{\mathrm{cusp}}(\mathrm{GL}_n(F)) = \{ \text{irreducible } \varphi \} / \sim$

$\left. \begin{array}{l} \pi_\varphi = \pi_1 \otimes \dots \otimes \pi_r \text{ cuspidal} \\ \text{irreducible of } M = \prod_{i=1}^r \mathrm{GL}_{n_i}(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \varphi: \mathcal{W}_F \rightarrow \mathrm{GL}_n(\mathbb{C}) \text{ semisimple} \\ \varphi = \bigoplus_{i=1}^r \varphi_i \text{ with } \varphi_i: \mathcal{W}_F \rightarrow \mathrm{GL}_{n_i}(\mathbb{C}) \text{ irreducible} \end{array} \right.$

But  $\mathrm{Ind}_{M, \rho}^G \pi_\varphi$  is not irreducible in general

# LLC for $G_n$ (Langlands-Rapoport-Stuhler, Harris-Taylor, Henniart)

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$$\begin{array}{ccc} \text{Irr}_{\mathbb{C}}(G_n(F)) & \xrightarrow{\sim} & \Phi(G_n(F)) = \left\{ \begin{array}{l} \varphi: W_F \times SL_2(\mathbb{C}) \rightarrow G_n(\mathbb{C}) \\ \varphi|_{W_F} \text{ semisimple, } \varphi|_{SL_2(\mathbb{C})} \text{ algebraic} \end{array} \right\} / \sim \\ \pi \longmapsto & & \varphi_{\pi} \end{array}$$

"natural" bijections — satisfying a long list of properties:

• Central character:  $\omega_{\pi} \leftrightarrow \det(\varphi_{\pi})$  under LCFT

• Twisting  $\varphi_{\pi \otimes \chi} = \varphi_{\pi} \otimes \chi$  for  $\chi$  character of  $\begin{cases} F^* \\ W_F \end{cases}$

• Contragredient  $\varphi_{\pi^{\vee}} = \varphi_{\pi}^{\vee}$

• L &  $\epsilon$ -factors  $L(\pi \times \pi', s) = L(\varphi_{\pi} \otimes \varphi_{\pi'}, s)$

$$\epsilon(\pi \times \pi', s, \psi) = \epsilon(\varphi_{\pi} \otimes \varphi_{\pi'}, s, \psi)$$

# LLC for $GL_n$

$$\text{Irr}_c(GL_n(F)) \begin{array}{c} \xleftrightarrow{\sim} \\ \xrightarrow{\pi} \end{array} \underline{\Phi}(GL_n(F))$$

$$\text{DS}_c(GL_n(F)) \xleftrightarrow{\sim} \underline{\Phi}^{\text{ds}}(GL_n(F)) = \{ \text{indecomposable } \varphi = \varphi_0 \otimes \text{St}_a \}$$

$$\text{Cusp}_c(GL_n(F)) \xleftrightarrow{\sim} \underline{\Phi}^{\text{cusp}}(GL_n(F)) = \{ \text{irreducible } \varphi: W_F \rightarrow GL_n(\mathbb{C}) \}$$

It is compatible with parabolic induction:

$$IF \quad \pi \hookrightarrow \underbrace{I_{M,F}^G(\pi_1 \otimes \dots \otimes \pi_r)}_{\text{normalized parabolic induction}} \quad \text{then}$$

$$\underbrace{\varrho_\pi|_{W_F}} = \varrho_{\pi_1} \oplus \dots \oplus \varrho_{\pi_r}$$

where  $W_F \hookrightarrow W_F \times SL_2(\mathbb{C})$

$$w \longmapsto \left( w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right)$$

# LLC for $GL_n$ : depth

$$\text{Irr}_c(GL_n(F)) \xleftrightarrow{\pi} \underline{\Phi}(GL_n(F))$$

$\pi \longmapsto \rho_\pi$

$\cup \quad \cup$

$$\text{Irr}_c^{\circ}(GL_n(F)) \xleftrightarrow{\quad} \underline{\Phi}^{\text{trivial}}(GL_n(F)) = \{ \varphi \text{ s.t. } \varphi|_{\mathcal{P}_F} \text{ is trivial} \}$$

depth 0 representations  $(\pi, V)$ : i.e.  $V^{K^{\text{ot}}} \neq 0$ , where  $K^{\text{ot}} = 1 + \text{Mat}_n(\mathfrak{p})$

More generally the depth is preserved by LLC for  $GL_n$ :

$$l(\pi) = \inf \left\{ r \geq 0 : V^{\mathcal{P}^{r+1}} \neq 0 \text{ for some parahoric } \mathcal{P} \right\}$$

$$l(\varphi) = \inf \{ r \geq 0 : \varphi|_{W_F^{\text{ot}}} \text{ is trivial} \}$$

filtration  
 with  $W_F^0 = I_F$ ,  $W_F^{\text{ot}} = \mathcal{P}_F$

## Restriction to inertia

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$$\begin{array}{ccc} \text{Cusp}_e(\text{GL}_n(F)) & \longleftrightarrow & \Phi^{\text{cusp}}(\text{GL}_n(F)) \\ \pi & \longmapsto & \varrho_\pi \end{array}$$

Then

- $\varrho_\pi|_{I_F} = \varrho_{\pi'}|_{I_F} \iff \pi = \pi' \otimes \chi, \quad \chi \text{ unramified}$

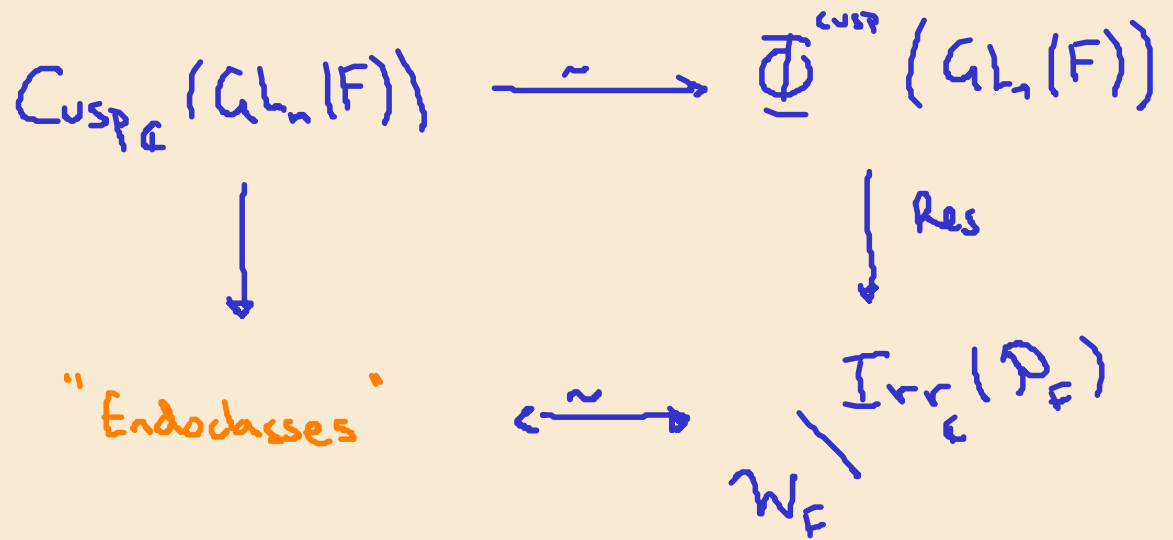
- (Paškūnas) Let  $\psi: I_F \rightarrow \text{GL}_n(\mathbb{C})$  be the restriction of some Langlands parameter.

There is a unique irreducible representation  $\lambda_\psi$  of  $\text{GL}_n(\mathcal{O}_F) = K$  such that

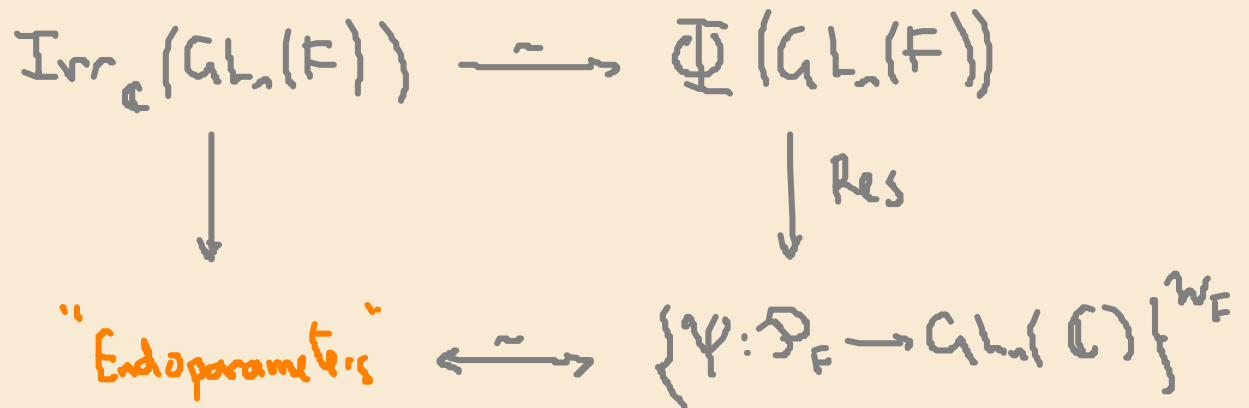
$$\text{Hom}_K(\lambda_\psi, \pi) \neq 0 \iff \varrho_\pi|_{I_F} = \psi$$

# Restriction to wild inertia (Bushnell - Henniart)

If  $\rho: W_F \rightarrow GL_n(\mathbb{C})$  is irreducible then  $\rho|_{\mathcal{P}_F}$  is a multiple of a  $W_F$ -orbit in  $\text{Irr}_c(\mathcal{P}_F)$ .



or





## Explicit LLC for $GL_n$ (Bushnell - Henniart)

$\varphi \in \underline{\mathbb{T}}^{\text{cusp}}(GL_n(F))$  is essentially tame if  $\varphi|_{\mathcal{P}_F}$  is a sum of characters

$(\pi, \nu) \in \text{Cusp}(GL_n(F))$  is essentially tame if the data involved in its construction is tame

$$\left[ \Leftrightarrow p \chi \frac{n}{t(\pi)} \quad \text{where } t(\pi) = \#\{\text{unramified } \chi : \pi \otimes \chi_{\text{det}} \simeq \pi\} \right]$$

Bushnell-Henniart make the correspondence explicit for  
essentially tame representations/parameters

## § 8 Local Langlands more generally

$G = G(F)$  has a complex dual group  $\hat{G} = \hat{G}(\mathbb{C})$

[Take based root datum for  $G$  and swap roots and coroots.]

|           |                    |                     |                         |                       |     |
|-----------|--------------------|---------------------|-------------------------|-----------------------|-----|
| $G$       | $GL_n(F)$          | $SL_n(F)$           | $Sp_{2n}(F)$            | $SO_{2n}(F)$          | ... |
| $\hat{G}$ | $GL_n(\mathbb{C})$ | $PGL_n(\mathbb{C})$ | $SO_{2n+1}(\mathbb{C})$ | $SO_{2n}(\mathbb{C})$ |     |

L-group  ${}^L G = \hat{G} \rtimes W_F$  (direct product for split groups)

Definition A parameter for  ${}^L G$  is continuous  $\varphi: W_F \times SL_2(\mathbb{C}) \rightarrow {}^L G$  s.t.

- $\varphi(w) \in \hat{G} w \quad \forall w \in W_F$
- $\varphi(w)$  is semisimple  $\forall w \in W_F$
- $\varphi|_{SL_2(\mathbb{C})}: SL_2(\mathbb{C}) \rightarrow \hat{G}$  is a homomorphism of complex alg. groups

$\Phi(G) = \{ \text{parameters for } {}^L G \text{ which are relevant for } G \} / \hat{G}\text{-conjugacy}$

LLC predicts

$$\text{Irr}_e(G) \longrightarrow \Phi(G)$$

with finite fibres (L-packets), satisfying various naturality conditions.

Warning L-packets can contain cuspidal and non-cuspidal representations  
(look at discrete series representations instead)

To parametrize representations in an L-packet, we need more:

Enhanced parameters  $(\varphi, \rho)$ , where  $\rho$  is an irreducible representation of some finite group  $S_\varphi$

Arthur There is such an LLC for symplectic and special orthogonal groups

# Semisimple LLC

Farques-Scholze give a map

$$\text{Irr}_c(G) \xrightarrow{\pi \mapsto \varphi_\pi^{\text{ss}}} \Phi^{\text{ss}}(G) = \{ \varphi: W_F \rightarrow {}^L G \} / \sim$$

with the expected properties, for arbitrary  $G$ .

Expect that

$$\varphi_\pi^{\text{ss}}(w) = \varphi_\pi \left( w \cdot \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix} \right)$$

## Hecke algebras (Arthur, Moussaoui, Solleveld)

$G = G(F)$  symplectic, (special) orthogonal, general (s)pin, unitary

$$s \in \mathcal{B}(G) \rightsquigarrow \text{block } \text{Rep}_c^s(G) \simeq \text{Mod-}\mathcal{H}_c^s$$

Also (for arbitrary  $G$ ) define cuspidal support of an enhanced Langlands parameter, and hence get Bernstein components  $\underline{\Phi}_e(G)^{s^v}$  and, for each such, build a Hecke algebra  $\mathcal{H}_c^{s^v}(z)$  ( $z$  an indeterminate) and

$$\underline{\Phi}_e(G)^{s^v} \longleftrightarrow \text{Irr}(\mathcal{H}_c^{s^v}(q^{\mathbb{N}}))$$

Match  $\text{Rep}_c^s(G)$  to  $\underline{\Phi}_e(G)^{s^v}$  using Arthur-Moeglin; then

$$(\mathcal{H}_c^s)^{\text{op}} \simeq \mathcal{H}_c^{s^v}(q^{\mathbb{N}})$$

Use this to build LLC

$$\text{Irr}(G) \longleftrightarrow \underline{\Phi}_e(G)$$

using only Arthur-Moeglin for cuspids

## §9 Endoclasses

The construction of cuspidal representations of  $G = GL_n(F)$  involves simple characters  $\Theta$ .

Data:  $E = F[\beta]$  extension of degree  $d$

$\varphi: E \hookrightarrow \text{End}_F(V)$  embedding

$x$  point in the Bruhat-Tits building  $\tilde{I}(G_\varphi, E) \hookrightarrow \tilde{I}(G, F)$

from which Bushnell-Kutzko defined

$\mathcal{L}(x, \varphi|\beta)$  a set of simple characters of a pro- $p$  group  $H^{\text{ot}}$

Remarkable Fact 1 There are canonical bijections  $\mathcal{L}(x, \varphi|\beta) \xrightarrow{\text{transfer}} \mathcal{L}(x', \varphi'|\beta)$

and  $\Theta \leftrightarrow \Theta'$  iff  $\underline{1}_\alpha$  intertwines  $\Theta$  with  $\Theta'$ .

But  $\Theta$  does not determine  $\varphi|\beta$ .

Suppose we have simple characters  $\theta \in \mathcal{L}(x, \varphi | \beta)$  and  $\theta' \in \mathcal{L}(x', \varphi' | \beta')$ .<sup>14</sup>

Definition  $\theta, \theta'$  are endo-equivalent if there exist  $\begin{cases} \varphi_i: F(\beta) \hookrightarrow \text{End}_F(V) \\ \varphi'_i: F(\beta') \hookrightarrow \end{cases}$   
and  $x_i, x'_i$  such

the transfers of  $\theta, \theta'$  to  $\mathcal{L}(x_i, \varphi_i | \beta), \mathcal{L}(x'_i, \varphi'_i | \beta')$  intertwine.

Remarkable Facts 2 (Bucknell-Henniart) Suppose  $\theta, \theta'$  are endo-equivalent.

- (i) The transfers of  $\theta, \theta'$  to any common space  $V$  intertwine.
- (ii) The transfers of  $\theta, \theta'$  to any common  $(V, x)$  are conjugate.

Definition An endo class is an endo-equivalence class of simple characters

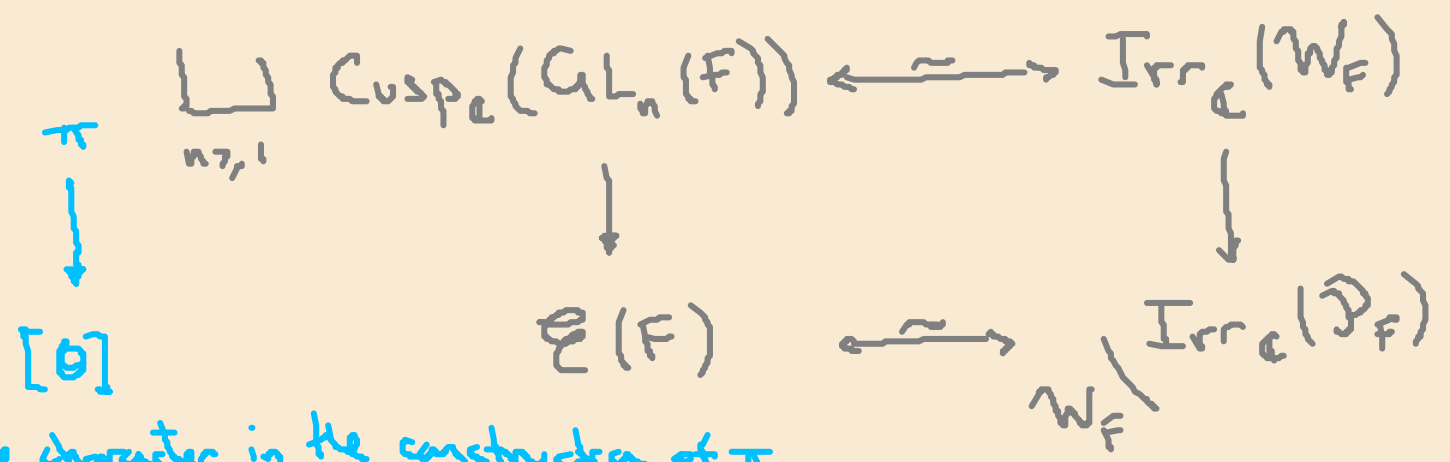
$\mathcal{E}(F)$  is the set of endo-classes

If  $(H) \in \mathcal{E}(F)$  contains a simple character  $\theta \in \mathcal{L}(x, \varphi | \beta)$  then

$$d_{\theta} = [F(\beta) : F], \quad e_{\theta} = e(F(\beta)/F), \quad f_{\theta} = f(F(\beta)/F)$$

depend only on  $(H)$ .

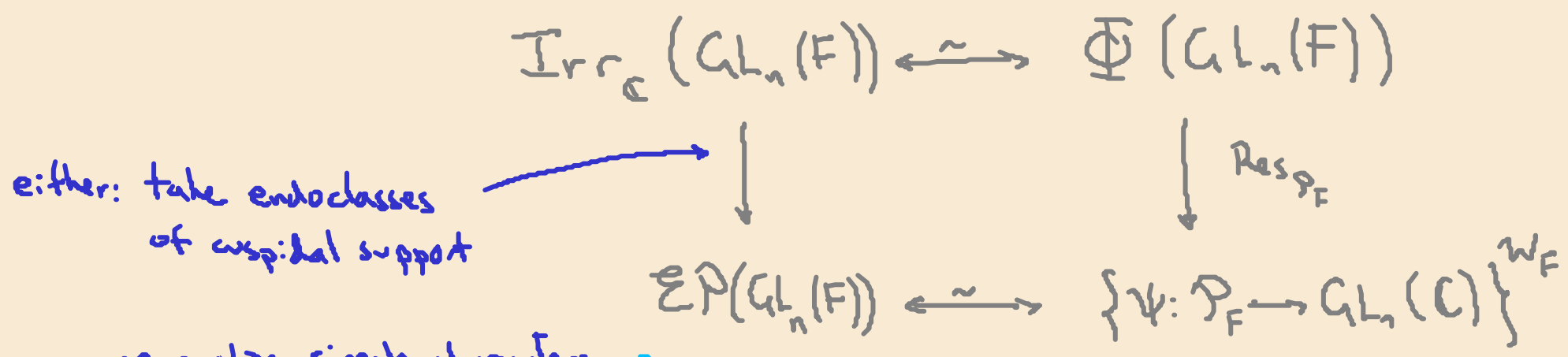
# Ramification Theorem (Bushnell - Henniart)



⊙ simple character in the construction of π

Definition  $\mathcal{EP}(\text{GL}_n(F)) = \left\{ (\oplus_i n_i \omega_i)_{i \in I} : \sum n_i d \omega_i = n \right\}$  degree n endo-parameters

Then



either: take endoclasses of cuspidal support

or: generalize simple characters to allow β to be semisimple

} then  $\mathcal{EP}(\text{GL}_n(F))$  is the intertwining classes of semi-simple characters



# Category decomposition (Vignéras, Helm-Kormiczuk-Shedlerack-S.)

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$$\text{Rep}_C(\text{GL}_n(F)) = \prod_{t \in \text{EP}(\text{GL}_n(F))} \text{Rep}_C^t(\text{GL}_n(F))$$

For suitable  $\Theta$  with endo-parameter  $t$ ,

$\text{ind}_{H^{\text{GL}_n(F)}}^{\text{GL}_n(F)} \Theta$  is a progenerator of  $\text{Rep}_C^t(\text{GL}_n(F))$

Remarks (1) This refines the depth decomposition  $\prod_{r \geq 0} \text{Rep}_C^r(\text{GL}_n(F))$ . (Vignéras)

(2) Only needs  $C$  to be a commutative  $\mathbb{Z}[\frac{1}{p}]$ -algebra.

(3) This generalizes to classical groups  $G$  ( $p \neq 2$ ) to give a set of endoparameters  $\text{EP}(G)$ . We expect

$$\text{EP}(G) \xleftrightarrow{\sim} \{\text{enhanced wild parameters for } G\}$$

# § 10 Other questions

Explicit LLC:

When can we make LLC explicit, with the data used to build representations?

- Depth zero DeBachter-Reeder
- Tame Katotha
- $\vdots$

Explicit transfer: If  $\mathcal{J}: H \rightarrow G_L$  then we should get

$$\begin{array}{ccc} \text{Irr}_e(H) & \xrightarrow{\text{LLC}} & \underline{\Phi}(H) \\ \downarrow \text{transfer} & & \downarrow \tau_0 \\ \text{Irr}_e(G_L(F)) & \xrightarrow{\text{LLC}} & \underline{\Phi}(G) \end{array}$$

Make this explicit.

Endoparameters for arbitrary G: are there such things which

- reflect restriction to wild inertia
- behave functorially in the group
- give a decomposition of  $\text{Rep}_c(G)$

Moreover, hopefully for  $\pm$  an endo-parameter

$$\text{Rep}_c^\pm(G) = \text{Rep}_c^\circ(G_\pm)$$

depth 0 reps of another group  $G_\pm$

[Chinello: true for  $\text{GL}_n(D)$ ]

Reduction to a unipotent block?

$$\text{Rep}_c^{s_u}(G) \simeq \text{Rep}_c^{s_u}(G_{s_u})$$

for some group  $G_{s_u}$  and unipotent inertial cuspidal support  $s_u$ .

## Other correspondences

Jacquet-Langlands: bijection between essentially square-integrable irreducible representations of  $G_h(F)$  and an inner form.

Explicit description? • Dotto does so at the inertial level  
(it preserves endo-classes)

Compatibility with mod- $\ell$  congruences / reduction modulo  $\ell$ : Minguez-Sécherre

Theta correspondence: bijection between certain representations of a dual pair  $(G, G')$

Explicit description? • Loke-Mao in the tame case

Compatibility with mod- $\ell$  congruences / reduction modulo  $\ell$ ?