On Hasse-Weil zeta functions of Kottwitz simple Shimura varieties

Thomas J. Haines

Mathematics Department
University of Maryland

Langlands program: Number Theory and Representation Theory
BIRS – Oaxaca
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Joint with Jingren Chi

I will report on joint work in progress with Jingren Chi.

Outline

1. Shimura varieties and statement of main theorem
2. The Test Function Conjecture
3. The Scholze Test functions
4. The new local ingredients

Shimura data

- $(\mathbf{G}, X)$: $\mathbf{G}$ connected reductive $\mathbb{Q}$-group, $X = \mathbf{G}(\mathbb{R})/K_\infty$ Hermitian symmetric space; these are required to satisfy the axioms of Deligne.
- In particular, $X \ni h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_\mathbb{R}$ gives rise to a $\mathbf{G}(\mathbb{C})$-conjugacy class $\{\mu\}$ of *minuscule* cocharacters $\mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_\mathbb{C}$ (the Shimura cocharacters).
- Example: $\mathbf{G} = \mathrm{GL}_2$, $X = \mathbb{C}\backslash\mathbb{R}$, $\mu(z) = \mathrm{diag}(z, 1)$, i.e., $\mu = (1, 0)$.
- Let $\mathbf{E} \subset \mathbb{C}$ be the field of definition of $\{\mu\}$. Then for $K \subset \mathbf{G}(\mathbb{A}_f)$ ranging over compact open subgroups,

$$\mathrm{Sh}(\mathbf{G}, X)(\mathbb{C}) := \varprojlim_K \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}) := \varprojlim_K \mathbf{G}(\mathbb{Q})\backslash[X \times \mathbf{G}(\mathbb{A}_f)/K],$$

  is the $\mathbb{C}$-points of inverse limit of quasi-projective varieties over $\mathbf{E}$.
- Abbreviate $\mathrm{Sh}_K := \mathrm{Sh}_K(\mathbf{G}, X)$.

Hecke correspondences and Galois actions

- Fix rational prime ℓ. Set $H^i = \varinjlim_K H^i_{\mathrm{et}}(\mathrm{Sh}_K \otimes_{\mathbf{E}} \bar{\mathbb{Q}}\,,\, \bar{\mathbb{Q}}_\ell)$.
- This is naturally a representation of $\mathbf{G}(\mathbb{A}_f) \times \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbf{E})$.
- Decompose the alternating sum H^* in the Grothendieck group

$$H^* = \sum_{\pi_f} \pi_f \otimes \sigma(\pi_f).$$

  where $\pi_f \in \mathrm{Irrep}(\mathbf{G}(\mathbb{A}_f))$ and $\sigma(\pi_f)$ is the associated (virtual) representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbf{E})$.
- The map $\pi_f \mapsto \sigma(\pi_f)$ is supposed to reflect aspects of the **global Langlands correspondence** for the group $\mathbf{G}$.
- Expect local-global compatibility: **roughly**, if $\pi_f = \pi^p \otimes \pi_p$ for $p \neq \ell$, then $\pi_p \in \mathrm{Irrep}(\mathbf{G}(\mathbb{Q}_p))$ should belong to the **local** L-**packet** given by $\pm\sigma(\pi_f)|_{W_{\mathbf{E}_\mathfrak{p}}}$ where $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_\mathbf{E})$ has $\mathfrak{p}|p$.
- First step: understand H^* as virtual $\mathbf{G}(\mathbb{A}_f) \times W_{\mathbf{E}_\mathfrak{p}}$-module, for the local Weil group $W_{\mathbf{E}_\mathfrak{p}} \subset \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbf{E}_\mathfrak{p})$.

Statement of the problem

- Write $G = \mathbf{G}_{\mathbb{Q}_p}$, $E = \mathbf{E}_{\mathfrak{p}}$, ${}^LG = \widehat{G}(\bar{\mathbb{Q}}_\ell) \rtimes W_{\mathbb{Q}_p}$, and ${}^LG_E = \widehat{G}(\bar{\mathbb{Q}}_\ell) \rtimes W_E$.
- The Shimura cocharacter $\mu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$ gives rise to an algebraic representation $r_{-\mu} : {}^LG \to \mathrm{Aut}(V_{-\mu})$ on the highest weight $\widehat{G}$-rep. $V_{-\mu}$.

Theorem (Target Theorem)

For "nice" unitary Shimura varieties (e.g., simple Kottwitz type, i.e.,compact, no global endoscopy), we have an isomorphism of virtual $\mathbf{G}(\mathbb{A}_f^p) \times K_p^0 \times W_E$-modules

$$H^* \cong \sum_{\pi_f = \pi^p \otimes \pi_p} a(\pi_f)\, \pi_f \otimes (r_{-\mu} \circ \varphi_{\pi_p}|_{W_E})|\cdot|^{-\dim \mathrm{Sh}/2}.$$

for certain "virtual multiplicities" $a(\pi_f) \in \mathbb{Z}$ defined by Kottwitz.

- Here $K_p^0 \subset G(\mathbb{Q}_p)$ is a compact open subgroup (for us, a parahoric), and $\varphi_{\pi_p} : W_{\mathbb{Q}_p} \to {}^LG$ is the semisimple local Langlands parameter attached to π_p.

Consequence for local Hasse-Weil zeta functions

Corollary

In the above situation, let $K \subset \mathbb{G}(\mathbb{A}_f)$ be any sufficiently small compact open subgroup. Then the semisimple local Hasse-Weil factor of Sh_K at the place $\mathfrak{p}$ of $\mathbf{E}$ is given by

$$\zeta^{\mathrm{ss}}_{\mathfrak{p}}(\mathrm{Sh}_K, s) = \prod_{\pi_f} L^{\mathrm{ss}}(s - \frac{\dim \mathrm{Sh}_K}{2}, \pi_p, r_{-\mu})^{a(\pi_f)\dim \pi_f^K}$$

- The Kottwitz simple Shimura varieties: let $F \supset F^+ \supset \mathbb{Q}$ be a CM field, let $(\mathbf{D}, *)$ be a central division algebra over F of rank n^2, with involution $*$ of 2nd type, let $\mathbf{G} = \mathrm{GU}(\mathbf{D}, *)$.
- We get a "fake" unitary group Shimura variety, which is projective, and has no global endoscopy.
- It is PEL type: for K_p parahoric $\mathrm{Sh}_{K^pK_p}$ has integral $\mathcal{O}_E$-model the moduli space of abelian schemes with additional structure $(A_\bullet, \lambda, \iota, \eta^p)$.
- The endomorphism structure $\iota : \mathcal{O}_{\mathbf{D}} \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ must satisfy Kottwitz' determinant condition, and η^p is a K^p-level structure on A.

Some History: Target Theorem proved by

- Kottwitz: in good reduction situation, where $G/\mathbb{Q}_p$ is unramified, $K_p^0 = \mathcal{G}(\mathbb{Z}_p)$ for a hyperspecial maximal parahoric group scheme $\mathcal{G}$. All π_p appearing are unramified.
- Harris-Taylor: in the Harris-Taylor cases. Among other things, essentially $\mathbf{G}_{\mathbb{R}} = \mathrm{GU}(1, n-1) \times \mathrm{GU}(0, n)^{[F^+:\mathbb{Q}]-1}$. That is, essentially $\mu = (1, n-1)$. And $G = \prod_i \mathrm{GL}_{n_i} \times \prod_j D_j^\times$.
- Scholze-Shin: like Kottwitz (any signature at ∞, any μ), but require $G = \mathbf{G}_{\mathbb{Q}_p}$ a product of Weil-restrictions of GL_n's.
- Xu Shen: like Harris-Taylor, essentially $\mu = (1, n-1)$, but G required to be a certain product of units of division algebras with invariants $\pm 1/n_i$ (p-adic uniformization situation).

Theorem (Chi-H.)

The Target theorem holds in the Scholze-Shin situation, except we may allow G to be any inner form of a product of Weil-restrictions of GL_n's.

- We thus generalize both Scholze-Shin and Shen, but our proof is different from either.

Counting Points

- We fix a parahoric $\mathcal{G}(\mathbb{Z}_p) = K_p^0$, and integral model $\mathcal{S}_{K_p^0} = \mathrm{Sh}_{K^p K_p^0}$ over $\mathcal{O}_E$. For $K_p \subset K_p^0$, we have finite étale $\pi_{K_p K_p^0} : \mathrm{Sh}_{K^p K_p} \to \mathrm{Sh}_{K^p K_p^0}$.
- From now on, fix sufficiently small $K = K^p K_p$ with $K_p \subset K_p^0$ arbitrarily deep.
- Let $f^p \in C_c^\infty(\mathbf{G}(\mathbb{A}_f^p)//K^p)$, $h \in C_c^\infty(K_p^0//K_p)$, $\tau \in \mathrm{Frob}^j I_E$, $r = j[\kappa_E : \mathbb{F}_p]$ (sufficiently large).
- Then Grothendieck-Lefschetz trace formula applied to Frobenius-Hecke correspondence **should** give a formula (for *some* function $\phi_{\tau,h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$)

$$\mathrm{Tr}(\tau \times hf^p \,|\, H^*) = \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta)\, O(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

- Point: RHS resembles the geometric side of the Arthur-Selberg trace formula for $\mathbf{G}$. So after (pseudo)stabilization, this would connect H^* with automorphic representations.

- Much work has shown: if K_p parahoric (and $h = e_{K_p}$), ϕ_τ is determined by $\mathrm{Tr}(\tau | R\Psi^{\mathcal{M}^{\mathrm{loc}}_{K_p}}(\bar{\mathbb{Q}}_\ell))$ (in center of a parahoric Hecke algebra).
- For deep K_p there is no integral model $\mathcal{S}_{K_p}$ and no local model $\mathcal{M}^{\mathrm{loc}}_{K_p}$, but nevertheless we still expect to be able to take a function in the **stable Bernstein center**.

Conjecture (The Test Function Conjecture, H.-Kottwitz)

For any Shimura variety, a formula like the above holds with $\phi_{\tau,h} = Z_{\tau,-\mu,r} \star \tilde{h}$. Here $\tilde{h} \in C_c^\infty(G(\mathbb{Q}_{p^r}))$ is any function with base-change transfer $h \in C_c^\infty(G(\mathbb{Q}_p))$.

- Here $Z_{\tau,-\mu,r}$ is an element in the usual Bernstein center $\mathfrak{Z}(G/\mathbb{Q}_{p^r})$ which is the image of an element $Z_{\tau,V_{-\mu,r}}$ in the stable Bernstein center $\mathfrak{Z}^{\mathrm{st}}(G/\mathbb{Q}_{p^r})$.

(Stable) Bernstein center

- Change notation: F non-arch. local, G connected reductive over F. Choose isom $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.
- The Bernstein variety: a variety structure on the set of supercuspidal supports $(M(F), \sigma)_G$. (Act by twisting σ by elements of the torus $X^{\mathrm{un}}(M) = \mathrm{Hom}(M(F)/M(F)^1, \mathbb{C}^\times)$.)
- The Bernstein center is the ring of regular functions $\mathfrak{Z}(G/F)$.
- The stable Bernstein variety: a variety structure on $\widehat{G}$-conjugacy classes of semisimple Langlands parameters $\varphi : W_F \to {}^LG$.
- If $\varphi : W_F \to {}^LG$ **factors minimally** through ${}^LM \subset {}^LG$, we can twist it by any 1-cocycle in the torus $H^1(\mathrm{Frob}_F, Z(\widehat{M})^{I_F})^\circ \cong X^{\mathrm{un}}(M)$.
- Stabilizers are finite, so get infinite union of tori mod finite groups. The **stable Bernstein center** is the ring of regular functions $\mathfrak{Z}^{\mathrm{st}}(G/F)$.

Theorem (Fargues-Scholze)

For every G/F, there is a semisimple local Langlands correspondence $\pi \mapsto \varphi_\pi$, which is compatible with unramified twists, with normalized parabolic induction, and which is suitably functorial. Consequently, there is a natural homomorphism of commutative rings $\mathfrak{Z}^{\mathrm{st}}(G/F) \to \mathfrak{Z}(G/F)$.

- In particular, given $Z \in \mathfrak{Z}^{\mathrm{st}}(G/F)$, we can take its image Z in $\mathfrak{Z}(G/F)$ (thus view Z as a distribution on $G(F)$).
- Given any $(V, r) \in \mathrm{Rep}({}^L G)$, we get $Z_{\tau,V} \in \mathfrak{Z}^{\mathrm{st}}(G/F)$ by

$$Z_{\tau,V}(\varphi) := \mathrm{tr}(\tau \,|\, r \circ \varphi).$$

- **Upshot:** The Test Function Conjecture is now unconditional, thanks to Fargues-Scholze. They provided exactly what was needed.

- Scholze constructed functions $\phi_{\tau,h}$ which satisfy the point counting formula, using deformations of p-divisible groups.
- This applies to PEL Shimura varieties (Alex Youcis made progress extending this to abelian type Shimura varieties).
- Recall we don't have an integral model at level K_p. Scholze suggested pushing down to the integral model $\mathcal{S}_{K_p^0}$ and studying the nearby cycles of the sheaf $\pi_{K_p^0 K_p,*}(\bar{\mathbb{Q}}_\ell)$ on the generic fiber $\mathrm{Sh}_{K_p^0}$.
- But he still needed a purely local construction of the test function $\phi_{\tau,h}$, which he defined as follows.
- Setting: $\kappa \supset \kappa_E$ perfect field of characteristic p, endowed with $\mathcal{O}_E \to \kappa$. Let $H_\bullet$ chain of p-divisible groups over κ. (Really, consider those with (P)EL structure: action of $\mathcal{O}_B$, Kottwitz determinant condition, etc.).

Definition

The deformation space $\mathcal{X}_{H_\bullet}$ of $H_\bullet$ is the functor that associates to any Artin local $\mathcal{O}_E$-algebra R with residue field κ the set of isomorphism classes of p-divisible groups (with (P)EL)-structure) $\tilde{H}_\bullet$ over $\mathrm{Spec}(R)$, together with an isomorphism $\tilde{H}_\bullet \otimes_R \kappa \xrightarrow{\sim} H_\bullet$, i.e. an isomorphism of p-divisible $\mathcal{O}_B$-modules $\tilde{H}_\bullet \otimes_R \kappa \xrightarrow{\sim} H_\bullet$.

Scholze test function, cont'd

- Scholze proved: The functor $\mathcal{X}_{H_\bullet}$ is represented by a complete Noetherian local $\mathcal{O}_E$-algebra $R_{H_\bullet}$ with residue field κ.
- Let $X_{H_\bullet}$ denote the Raynaud generic fiber of the formal scheme $\mathrm{Spf}(R_{H_\bullet})$. This is a rigid analytic space over $k := W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}]$.
- If chain $H_\bullet$ corresponds to parahoric $\mathcal{G}(\mathbb{Z}_p)$, any $K \subset \mathcal{G}(\mathbb{Z}_p)$ gives étale cover $\pi_K : X_{H_\bullet, K} \to X_{H_\bullet}$ (parametrize level-K structures on universal p-adic Tate module).
- Rational covariant Dieudonné module $M_\bullet \otimes \mathbb{Q}$ with σ-linear Frobenius F can be rigidified:

$$(M_\bullet \otimes \mathbb{Q}, F) \cong (\Lambda_\bullet \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, p\delta\sigma),$$

  for $\delta \in G(W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}])$, well-defined up to $G(W_{\mathcal{O}_E}(\kappa))$-$\sigma$-conjugacy.
- The association $H_\bullet \mapsto \delta \in G(W_{\mathcal{O}_E}(\kappa)[\frac{1}{p}])/\sim$ is an injection.

End of Scholze construction

Definition

Let $\delta \in G(\mathbb{Q}_{p^r})$. If δ comes from an $H_\bullet$ over $\kappa = \mathbb{F}_{p^r}$ with controlled cohomology, then set

$$\phi_{\tau,h}(\delta) = \mathrm{tr}(\tau \times h \,|\, H^*(X_{H_\bullet,K} \otimes_k \hat{\bar{k}}, \mathbb{Q}_\ell)),$$

where K is any normal pro-p open compact subgroup such that h is K-biinvariant. If δ does not arise this way, set $\phi_{\tau,h}(\delta) = 0$.

- Scholze: the function $\phi_{\tau,h}$ is locally constant $\mathbb{Q}$-valued, compactly supported function on $G(\mathbb{Q}_{p^r})$. It is $\mathcal{G}(\mathbb{Z}_{p^r})$-$\sigma$-conjugacy invariant. It is independent of $\ell \neq p$.
- His main result: This function $\phi_{\tau,h}$ satisfies the point counting formula (PEL type cases):

$$\mathrm{Tr}(\tau \times hf^p \,|\, H^*) = \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta)\, O(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

Wrinkle

$$\mathrm{Tr}(\tau \times hf^p \,|\, H^*) = \sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta)\, O(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

- Small lie: Scholze requires $G/\mathbb{Q}_p$ *unramified*, in particular, quasi-split.
- In general, there is an obstruction to constructing the triple (γ_0,γ,δ) from a Frobenius-Hecke fixed point: if G is **not quasi-split**, then there is no reason that $N_r(\delta) := \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ should be stably conjugate to an element in $G(\mathbb{Q}_p)$.
- This is the main source of difficulty we encounter for non-quasi-split groups.
- To make things work, we need to know the **a priori vanishing statement**: $TO_{\delta\sigma}(\phi_{\tau,h}) = 0$ if $N_r(\delta)$ is not stably conjuate to $G(\mathbb{Q}_p)$.
- Another question: How is Scholze's $\phi_{\tau,h}$ related to the conjectural test functions $Z_{\tau,-\mu,r} \star \tilde{h}$?

Relation between test functions

- How is Scholze's $\phi_{\tau,h}$ related to the conjectural test function $Z_{\tau,-\mu,r} \star \tilde{h}$?

Theorem (Chi-H.)

In the various situations of the Target Theorem, the two test functions essentially agree, in the sense that for any $\delta \in G(\mathbb{Q}_{p^r})$ with $N_r(\delta)$ semisimple, we have $TO_{\delta\sigma}(\phi_{\tau,h}) = TO_{\delta\sigma}(Z_{\tau,-\mu,r} \star \tilde{h})$.

- Thus, in light of Scholze point-counting formula, the Test Function Conjecture holds here.

- Our main local results are the following.

Theorem (Chi-H.)

In the situation of the Target Theorem,

(1) *The Scholze function $\phi_{\tau,h}$ satisfies the a priori vanishing property. Thus it has a base-change to a function $b(\phi_{\tau,h}) \in C_c^\infty(G(\mathbb{Q}_p))$.*

(2) *We have $b(\phi_{\tau,h}) = Z_{\tau,-\mu,1} \star h$.*

- The unusual notion of base-change in (1) appears because we need to use **pseudo-stabilization** to rewrite the point-counting formula in terms of automorphic representations for $\mathbf{G}$.
- This is proved by a global method. (1) is proved first: by embedding the local situation into that attached to a certain global group $\mathbb{G}$, and using Harris-Taylor and Fargues methods, we can arrange

$$\mathrm{tr}(\tau \times hf^p | H^*_{\mathbb{G}}) = \sum_{\pi} m(\pi) \mathrm{Tr}(f_{\tau,h} f^p f^{\mathbb{G}}_\infty \,|\, \pi)$$

where $f_{\tau,h} := Z_{\tau,-\mu,1} \star h$ and the sum ranges over automorphic representations of $\mathbb{G}$, but with original objects at p.

- The stable Bernstein center at p does not change when $\mathbb{G}$ is replaced by a global inner form $\mathbb{G}'$ which is isomorphic to $\mathbb{G}$ outside p, ∞, but which is quasi-split at p.
- For a well-chosen $\mathbb{G}'$, we can use global results for the simple Kottwitz Shimura variety attached to $\mathbb{G}'$. Recall $\mathbb{G}'_{\mathbb{Q}_p}$ is quasi-split by assumption. Namely we have

$$\mathrm{tr}(\tau \times hf^p \,|\, H^*_{\mathbb{G}'}) = \sum_{(\gamma_0', \gamma, \delta)} c(\gamma_0', \gamma\delta)\, O_\gamma(f^p)\, TO_{\delta\sigma}(\phi_{\tau,h}).$$

  where $(\gamma_0', \gamma, \delta)$ ranges over generalized Kottwitz triples such that $\gamma_0' \in \mathbb{G}'(\mathbb{Q})$ is transfered from $\delta \in \mathbb{G}(\mathbb{Q}_{p^r}) = G(\mathbb{Q}_{p^r})$.
- The RHS is $\sum_{\pi'} m(\pi')\mathrm{tr}\,\pi'(f^*_{\tau,h} f^p f^{\mathbb{G}'}_\infty)$, where the sum if over automorphic representations π' of $\mathbb{G}'$ and $f^*_{\tau,h} \in \mathbb{G}'(\mathbb{Q}_p)$ is the twisted endoscopic transfer of $\phi_{\tau,h} \in \mathbb{G}(\mathbb{Q}_{p^r})$.
- Then a comparison of global trace formulas for $\mathbb{G}$ and $\mathbb{G}'$ allows us to deduce (1,2) for $\mathbb{G}$. The Target Theorem follows.