Cohomology of stacks of shtukas I

§1. Introduction

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$, $\mathrm{char}(\mathbb{F}_q) = p$.

Let F be the function field of X.

Let G be a (split) connected reductive group over $\mathbb{F}_q$.

For simplicity, we assume that G is semisimple and we consider the case without level structure.

Let $\ell \neq p$. Let $\hat{G}$ be the Langlands dual group of G over $\overline{\mathbb{Q}}_\ell$.

e.g. $G = GL_n$, $\hat{G} = GL_n$

$G = SL_n$, $\hat{G} = PGL_n$

$G = SO_{2n+1}$, $\hat{G} = Sp_{2n}$

Automorphic side:

Let $\mathbb{A}$ be the ring of adèles of F

$\mathbb{O}$ be the ring of integral adèles.

We have the vector space of automorphic forms:

$$\mathrm{Funct}_c\left(G(F)\backslash G(\mathbb{A}) / G(\mathbb{O}), \overline{\mathbb{Q}}_\ell \right)$$

(continuous functions with compact support)

It is a $\overline{\mathbb{Q}}_\ell$-vector space, may have infinite dimension.

It is equiped with an action of the global Hecke algebra $\mathcal{H}_G := \mathrm{Funct}_c\left(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \overline{\mathbb{Q}}_\ell\right)$ by convolution.

Let Bun_G be the classifying stack of G-bundles:

for S affine scheme over $\mathbb{F}_q$,

$\mathrm{Bun}_G(S) = \{ G\text{-bundles over } X \times S \}$

all the products are over $\mathbb{F}_q$

$X \times S := X \underset{\mathbb{F}_q}{\times} S$

It is an Artin stack over $\mathbb{F}_q$.

Exemple: $G = SL_n$, $\mathrm{Bun}_{SL_n}(S) = \{$ rank n vector bundles over $X \times S$, with a trivialization of determinant $\}$

in particular, if $X = \mathbb{P}^1$, $G = SL_2$

$$\mathrm{Bun}_{SL_2}(\overline{\mathbb{F}}_q) = \{ \mathcal{O}(n) \oplus \mathcal{O}(-n),\ n \in \mathbb{Z}_{\geq 0} \}$$

Fact: we have an isomorphism

$$G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) \simeq \mathrm{Bun}_G(\mathbb{F}_q)$$

Thus the space of automorphic forms is

$$\mathrm{Funct}_c\left(\mathrm{Bun}_G(\mathbb{F}_q),\ \overline{\mathbb{Q}}_\ell\right)$$

§ 2. Stacks of shtukas

Let $I = \{1, 2, \cdots, n\}$ be a finite set.

We define the Hecke stack $\boxed{\mathrm{Hecke}_{G,I}}$:

for S affine scheme over $\mathbb{F}_q$,

$$\mathrm{Hecke}_{G,I}(S) := \left\{ (x_1, \cdots, x_n),\ \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} \mathcal{E}_n \right\}$$

where $x_1, \cdots, x_n \in X(S)$

$\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n \in \mathrm{Bun}_G(S)$ $\qquad \Gamma_{x_1}$: graph of x_1 in $X \times S$

$\mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1$ is an isomorphism over $X \times S - \Gamma_{x_1}$

$\mathcal{E}_1 \overset{\phi_2}{\dashrightarrow} \mathcal{E}_2$ is an isomorphism over $X \times S - \Gamma_{x_2}$

$\cdots$

$\mathcal{E}_{n-1} \overset{\phi_n}{\dashrightarrow} \mathcal{E}_n$ is an isomorphism over $X \times S - \Gamma_{x_n}$

$$\begin{tikzcd} Sht_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Hecke_{G,I} \arrow[d, "(pr_0, pr_n)"] & \left((x_1,\cdots,x_n), \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_n}{\dashrightarrow} \mathcal{E}_n\right) \arrow[d, mapsto] \\ Bun_G \arrow[r, "(Id, Frob)"] & Bun_G \times Bun_G & (\mathcal{E}_0, \mathcal{E}_n) \end{tikzcd}$$

Recall that Bun_G is over $\mathbb{F}_q$. We have the Frobenius morphism $Bun_G \xrightarrow{Frob} Bun_G$.

For S affine scheme over $\mathbb{F}_q$,

$$\begin{tikzcd} S \arrow[r] \arrow[d, "Frob"'] \arrow[dr, phantom, "\circlearrowright"] & Bun_G \arrow[d, "Frob"] \\ S \arrow[r] & Bun_G \end{tikzcd}$$

$$Bun_G(S) \xrightarrow{Frob} Bun_G(S)$$

$$\mathcal{E} \text{ over } X \times S \longmapsto (Id_X \times Frob_S)^* \mathcal{E} =: {}^\tau\mathcal{E}$$

We define the <u>stack of shtukas</u> with I paws to be the fiber product above Sht_I.

We call <u>a shtuka</u> a S-point of Sht_I.

$x_i \in X(S)$

$$\mathrm{Sht}_I(S) = \left\{ (x_1, \cdots, x_n),\ \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_{n-1} \overset{\phi_n}{\dashrightarrow} \mathcal{E}_n \simeq {}^{\tau}\mathcal{E}_0 \right\}$$

$$= \left\{ (x_1, \cdots, x_n),\ \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_{n-1} \overset{\phi_n}{\dashrightarrow} {}^{\tau}\mathcal{E}_0 \right\}$$

Fact: Sht_I is an inductive limit of Deligne-Mumford stacks.

Exemple: when $I = \phi$ empty set, $\mathrm{Hecke}_{G,I} = \mathrm{Bun}_G$

the above diagram is

$$\begin{tikzcd}
\mathrm{Bun}_G(\mathbb{F}_q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Bun}_G \arrow[d, "\Delta"] \\
\mathrm{Bun}_G \arrow[r, "(\mathrm{Id}, \mathrm{Frob})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G
\end{tikzcd}$$

$$\mathrm{Sht}_I = \mathrm{Bun}_G(\mathbb{F}_q)$$

§ 3 Geometric Satake equivalence

Let Gr_G be the affine grassmanian, i.e. $\mathrm{Gr}_G = LG / L^+G$,

where: loop group $LG(\mathrm{Spec} R) = G(R((t)))$

positive loop group $L^+G(\mathrm{Spec} R) = G(R[[t]])$

Fact: Gr_G is an ind-scheme over $\overline{\mathbb{F}_q}$.

For $S = \mathrm{Spec}\, R$ affine scheme over $\overline{\mathbb{F}_q}$,

$$Gr_G(S) = \left\{ \mathcal{E}_0 \overset{\phi}{\dashrightarrow} \mathcal{E}_1 \overset{\theta}{\simeq} \underset{\mathcal{E}_{triv}}{\cancel{G \times \mathrm{Spec}\, R[[t]]}} \right\}$$

where $\mathcal{E}_0, \mathcal{E}_1$ are G-bundles over $\mathrm{Spec}\, R[[t]]$

ϕ is an isomorphism over $\mathrm{Spec}\, R((t))$

θ is a trivialization over $\mathrm{Spec}\, R[[t]]$

$L^+G \circlearrowright Gr_G$

Geometric Satake equivalence:

$$\mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(\hat{G}) \simeq \mathrm{Perv}_{L^+G}(Gr_G)$$

$W \mapsto \mathcal{S}_W$ supported on some closed subscheme of finite dimension $Gr_{G,W}$

tensor product — fusion product

Functorial:

$W_1 \oplus W_2 \mapsto \mathcal{S}_{W_1} \oplus \mathcal{S}_{W_2}$ support on $Gr_{G,W_1} \cup Gr_{G,W_2}$

$W_1 \longrightarrow W_2 \mapsto \mathcal{S}_{W_1} \rightarrow \mathcal{S}_{W_2}$

When W is irreducible, $\mathcal{S}_W$ is isomorphic to $\underset{\substack{\downarrow\\ \text{intersection complex}}}{IC}_{Gr_G, W}$.

Now let $\overset{\{1,2,\cdots,n\}}{\overset{\shortparallel}{I}}$ be a finite set.

Let $Gr_{G,I}$ be the Beilinson-Drinfeld affine grassmanian

over X^I.

$$Gr_{G,I}(S) = \left\{ (x_1, \cdots, x_n),\ \mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1 \dashrightarrow \cdots \overset{\phi_n}{\dashrightarrow} \mathcal{E}_n \overset{\theta}{\underset{\sim}{\to}} \mathcal{E}_{triv} \right\}$$

where $x_1, \cdots, x_n \in X(S)$

$\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n$ are G-bundles over $\underbrace{\bigcup_{i \in I} \Gamma_{\infty x_i}}$

union of formal completions of graph of x_i

$\mathcal{E}_0 \overset{\phi_1}{\dashrightarrow} \mathcal{E}_1$ is an isom over $\bigcup_{i \in I} \Gamma_{\infty x_i} - \Gamma_{x_1}$

...

$\mathcal{E}_{n-1} \overset{\phi_n}{\dashrightarrow} \mathcal{E}_n$ is an isom over $\bigcup_{i \in I} \Gamma_{\infty x_i} - \Gamma_{x_n}$

θ is a trivialization over $\bigcup_{i \in I} \Gamma_{\infty x_i}$

Let $(L^+G)_I$ be the family of positive loop groups over X^I.

It acts on $Gr_{G,I}$.

We have a functor of monoidal categories:

$$\begin{array}{ccc} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}^I) & \longrightarrow & \mathrm{Perv}_{(L^+G)_I}(Gr_{G,I}) \\ W & \longmapsto & \mathcal{S}_{I,W} \end{array}$$

supported on some closed subscheme of finite dim $Gr_{G,I,W}$

Functorial

Factorisation structure.

§4. ~~Como~~ Cohomology of stacks of shtukas

We have a morphism by restriction:

$$\mathrm{Hecke}_{G,I} \longrightarrow [(L^+G)_I \backslash Gr_{G,I}]$$

$$\left((x_1,\dots,x_n), \mathcal{E}_0 \xrightarrow{\phi_1} \mathcal{E}_1 \dashrightarrow \cdots \xrightarrow{\phi_n} \mathcal{E}_n\right) \longmapsto \left((x_1,\dots,x_n), \mathcal{E}_0|_{\cup\Gamma_{\infty x_i}} \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_n} \mathcal{E}_n|_{\cup\Gamma_{\infty x_i}}\right)$$

$$\begin{tikzcd} Sht_I \arrow[r] \arrow[rr, bend right, "\mathcal{E}"'] \arrow[dr] & \mathrm{Hecke}_{G,I} \arrow[r] \arrow[d] & {[(L^+G)_I \backslash Gr_{G,I}]} \arrow[dl] \\ & X^I & \end{tikzcd}$$

Proposition (V. Lafforgue): $\mathcal{E}$ is formally smooth.

For $W \in Rep_{\bar{\mathbb{Q}}_\ell}(\hat{G}^I)$, we define $\mathcal{F}_{I,W} := \mathcal{E}^* \mathcal{S}_{I,W}$.

$\mathcal{F}_{I,W}$ is a perverse sheaf on Sht_I, supported on some finite dim closed substack $Sht_{I,W}$.

When W is irreducible, $\mathcal{F}_{I,W}$ is isomorphic to $IC_{Sht_{I,W}}$.

$$\begin{array}{ll} Sht_I & \mathcal{F}_{I,W} \\ \big\downarrow p & \\ X^I & \mathcal{H}_{I,W} := Rp_! \mathcal{F}_{I,W} \end{array}$$

Def: $\forall j \in \mathbb{Z}$, we define the degre j cohomology sheaf

$$\mathcal{H}^j_{I,W} := R^j p_! \mathcal{F}_{I,W}$$

It is an inductive limit of constructible $\bar{\mathbb{Q}}_\ell$-sheaves over X^I.

In fact, we have the Harder-Narasimhan stratification:

$$\underbrace{\mathrm{Bun}_G}_{\text{loc. f.t.}} = \bigcup_{\substack{\mu \text{ dominant} \\ \text{coweight of } G}} \underbrace{\mathrm{Bun}_G^{\leq\mu}}_{\text{open substack of finite type}}$$

e.g. $G = SL_n$, $\mathrm{Bun}_{SL_n}^{\leq\mu}(S) = \{$ rk n v.b. $\mathcal{E}$ over $X\times S$, det trivial

$\forall$ geo pt $\bar{s}$ of S, the canonical HN filtration of

$\mathcal{E}|_{X\times\bar{s}}$ is bounded by μ $\}$

$X = \mathbb{P}^1$, $G = SL_2$, $\mu = (m, -m)$, $m \in \mathbb{Z}_{\geq 0}$, $\mathrm{Bun}_{SL_2}^{\leq(m,-m)}(\bar{\mathbb{F}}_q) = \{\mathcal{O}(n)\oplus\mathcal{O}(-n),\ n\leq m\}$

$$\begin{array}{ccc} \mathrm{Sht}_I & & ((x_1,\cdots,x_n),\ \mathcal{E}_0\to\mathcal{E}_1\to\cdots\to\mathcal{E}_n\simeq{}^\tau\mathcal{E}_0) \\ \downarrow & & \rotatebox{90}{$\mapsto$} \\ \mathrm{Bun}_G & & \mathcal{E}_0 \end{array}$$

$\mathrm{Sht}_I^{\leq\mu}$:= inverse image of $\mathrm{Bun}_G^{\leq\mu}$

$\mathrm{Sht}_I = \bigcup_\mu \mathrm{Sht}_I^{\leq\mu}$ $\quad$ $\mathrm{Sht}_{I,W} = \bigcup_\mu \underbrace{\mathrm{Sht}_{I,W}^{\leq\mu}}_{\text{finite type}}$

$\mathcal{H}_{I,W}^j = \varinjlim_\mu \mathcal{H}_{I,W}^{j,\leq\mu}$ $\quad$ $\mathcal{H}_{I,W}^{j,\leq\mu} := \cancel{H_c^j(\mathrm{Sht}_I^{\leq\mu}, \mathcal{F}_W)}$ $R^j p_!(\mathcal{F}_{I,W}|_{\mathrm{Sht}_{I,W}^{\leq\mu}})$

$\bar{\mathbb{Q}}_\ell$-constructible sheaf over X^I

when I = empty set, $\mathcal{H}_{I,W} = C_c^{\text{Funct}}(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$

W = trivial rep

Let $\eta = \text{Spec}\, F$ be the generic pt of X

$\bar{\eta} = \text{Spec}\, \bar{F}$ a geo pt over η

$\eta_I = \text{Spec}\, F_I$ the generic pt of X^I

$\overline{\eta_I} = \text{Spec}\, \overline{F_I}$ a geo pt over η_I

$$\begin{tikzcd}
\overline{\eta_I} \arrow[r] \arrow[d] & \eta_I \arrow[r] \arrow[d] & X^I \arrow[d, "\text{projection } pr_i"] \\
\bar{\eta} \arrow[r] & \eta \arrow[r] & X
\end{tikzcd}$$

cohomology group

$$\mathcal{H}^j_{I,W}\big|_{\overline{\eta_I}} = \varinjlim_{\mu} \underbrace{\mathcal{H}^{j,\le\mu}_{I,W}\big|_{\overline{\eta_I}}}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$H^j_c(\text{Sht}_I\big|_{\overline{\eta_I}}, \mathcal{F}_{I,W}) \qquad H^j_c(\text{Sht}_I^{\le\mu}\big|_{\overline{\eta_I}}, \mathcal{F}_{I,W})$$

$\overline{\mathbb{Q}}_\ell$- v.s. may have ∞ dim

It is equiped with

- ① an action of the Hecke alg $\mathcal{H}_G$
- ② an action of the partial Frobenius morphisms
- ③ an action of $\pi_1(\eta_I, \bar{\eta}_I) = Gal(\bar{F_I}/F_I)$

  (an action of $Weil(\bar{F_I}/F_I) \subset Gal(\bar{F_I}/F_I)$)

We will see that

$$② + ③ + \text{finiteness resules} \xrightarrow{\text{Drinfeld's lemma}} \text{an action of } Weil(\bar{F}/F)^I$$