LECTURE 2: QUADRATIC INTERSECTION THEORY

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Abstract. We introduce some basic notions about a quadratic refinement of intersection theory and characteristic classes.

1. Introduction

We have seen that the Chow groups, with their intersection product and the Chern classes of vector bundles, gives a path to computing enumerative invariants for geometric problems over an algebraically closed field. Here we refine this to a setting where the invariants live in the Grothendieck-Witt ring. This gives information on enumerative problems over the reals by taking the signature, and other invariants of quadratic forms, such as the discriminant, gives information over other fields.

2. Chow-Witt groups and Witt sheaf cohomology

There is a rather sophisticated description of the Chow ring of a smooth variety X as sheaf cohomology:

$$\mathrm{CH}^n(X) = H^n(X, \mathcal{K}^M_n) \tag{2.1}$$

where $\mathcal{K}^M_*$ is the sheaf of *Milnor K-groups*. For a local ring R (with infinite residue field), $K^M_*(R)$ is the tensor algebra on the group of units $R^\times$ modulo the Steinberg relation

$$K^M_*(R) := (R^\times)^{\otimes_{\mathbb{Z}}}/\langle u \otimes 1-u \mid u, 1-u \in R^\times\rangle$$

$K^M_*(R) = \oplus_{n\ge 0} K^M_n(R)$ is a graded ring with multiplication induced from the multiplication in the tensor algebra and extends to a sheaf of graded rings $\mathcal{K}^M_*$ on a scheme X with stalk at $x \in X$ $K^M_*(\mathcal{O}_{X,x})$; note that $\mathcal{K}^M_1 = \mathcal{O}_X^\times$ and $\mathcal{K}^M_0$ is the constant sheaf $\mathbb{Z}$. The identity (2.1) is known as *Bloch's formula*; this is the classical identity

$$H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X) = \mathrm{CH}^1(X)$$

for $n = 1$, and was proven in general by Kato. The main point is to show that $\mathcal{K}^M_n$ admits a flasque resolution of the form

$$0 \to \mathcal{K}^M_n \to \oplus_{x\in X^{(0)}} i_{x*}K^M_n(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(1)}} i_{x*}K^M_{n-1}(k(x)) \xrightarrow{\partial} \ldots$$
$$\xrightarrow{\partial} \oplus_{x\in X^{(n-1)}} i_{x*}K^M_1(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(n)}} i_{x*}K^M_0(k(x)) \to 0$$

Date: September 20, 2022.

with $X^{(q)}$ the set of codimension q points of X, so

$$\begin{aligned} H^n(X,\mathcal{K}^M_n) &= \mathrm{coker}[\oplus_{x\in X^{(n-1)}}K^M_1(k(x)) \xrightarrow{\partial} \oplus_{x\in X^{(n)}}K^M_0(k(x))] \\ &= \mathrm{coker}[\oplus_{x\in X^{(n-1)}}k(x)^\times \xrightarrow{\mathrm{div}} \oplus_{x\in X^{(n)}}\mathbb{Z}] \\ &= \mathrm{CH}^n(X). \end{aligned}$$

The quadratic refinement, the *Chow-Witt groups*, were first defined by Barge and Morel. Later one, Hopkins and Morel defined the *Milnor-Witt K-groups*, which lead to a definition of the Chow-Witt groups completely parallel to Bloch's formula.

For a field F, $K^{MW}_*(F)$ is the graded, associative $\mathbb{Z}$-algebra defined by generators and relations

- Generators:
  - $[u]$ in degree 1 for $u\in F^\times$
  - η in degree -1.
- Relations:
  - $[u]\eta=\eta[u]$ for all $u\in F^\times$
  - $[u][1-u]=0$ for $u, 1-u\in F^\times$
  - $[uv]=[u]+[v]+\eta[u][v]$
  - let $h:=2+\eta[-1]$. Then $\eta\cdot h=0$

Morel shows that the $K^{MW}_*(F)$ extend to define a sheaf of graded rings $\mathcal{K}^{MW}_*$ on a smooth k-scheme X. Here is a resumé of some of the first properties of this construction.

Proposition 2.1. *Let X be a smooth k-scheme.*

1. Let $\mathcal{GW}$, $\mathcal{W}$ denote sheaves of Grothendieck-Witt rings, resp. Witt groups, on X. There is natural isomorphism $\mathcal{K}^{MW}_0\cong\mathcal{GW}$ and for $n<0$ a natural isomorphism $\mathcal{K}^{MW}_n\cong\mathcal{W}$.

2. The element η defines a global section of $\mathcal{K}^{MW}_{-1}$ and $\mathcal{K}^{MW}_*/(\eta)\cong\mathcal{K}^M_*$.

3. Let $\mathcal{I}\subset\mathcal{GW}$ be the kernel of the rank homomorphism. Then for all $n\geq 0$, the surjection $\mathcal{K}^{MW}_n\to\mathcal{K}^M_n$ has kernel $\mathcal{I}^{n+1}$.

4. The assignment $X\mapsto\mathcal{K}^{MW}_{n,X}$ extends to a sheaf on smooth k-schemes: Let $f:Y\to X$ be a morphism of smooth k-schemes. There is a natural pullback map of sheaves $f^*: f^{-1}\mathcal{K}^{MW}_{n,X}\to\mathcal{K}^{MW}_{n,Y}$, with $(fg)^*=g^*f^*$. The items (1)-(3) are natural with respect to f^*.

Definition 2.2. Let X be a smooth k-scheme. For $n\geq 0$, the nth Chow-Witt group $\widetilde{\mathrm{CH}}^n(X)$ is defined as

$$\widetilde{\mathrm{CH}}^n(X) := H^n(X,\mathcal{K}^{MW}_n)$$

Via the surjection $\mathcal{K}^{MW}_n\to\mathcal{K}^M_n$, we have the map $\widetilde{\mathrm{CH}}^n(X)\to\mathrm{CH}^n(X)$, with kernel and cokernel arising from $H^*(X,\mathcal{I}^{n+1})$, which gives the new "quadratic" information. The pullback maps f^* for $f:Y\to X$ induces pullbacks $f^*:\widetilde{\mathrm{CH}}^n(X)\to\widetilde{\mathrm{CH}}^n(Y)$ compatible with the pullbacks $f^*:\mathrm{CH}^n(X)\to\mathrm{CH}^n(Y)$. There are also pushforward maps for proper maps, but here we need to introduce a new ingredient: *orientations* and *twisting*.

Given an invertible sheaf $\mathcal{L}$ on X, we can form the twisted version $\mathcal{GW}(L)$ of $\mathcal{GW}$, this being the sheaf of quadratic forms with values in $\mathcal{L}$ (instead of in $\mathcal{O}_X$).

$\mathcal{GW}(L)$ is a $\mathcal{GW} = \mathcal{K}_0^{MW}$ module by multiplication, and we can define the twisted Milnor-Witt sheaf by

$$\mathcal{K}_n^{MW}(\mathcal{L}) = \mathcal{K}_n^{MW} \otimes_{\mathcal{GW}} \mathcal{GW}(\mathcal{L})$$

We can think of a section of $\mathcal{K}_n^{MW}(L)$ as locally in the form $s \cdot \lambda$, with s a section of $\mathcal{K}_n^{MW}$ and λ a nowhere zero section of L, with the relation

$$s \cdot (u\lambda) = (\langle u \rangle \cdot s) \cdot \lambda$$

for u a unit.

Definition 2.3. The $\mathcal{L}$-twisted Chow-Witt groups are defined by

$$\mathrm{CH}^n(X; \mathcal{L}) := H^n(X, \mathcal{K}_n^{MW}(\mathcal{L}))$$

There is a Gersten-type resolution of the Milnor-Witt sheaves, which gives an interpretation of the Chow-Witt groups as "cycles with coefficients in the Grothendieck-Witt group". This is called the *Rost-Schmid resolution* and looks like this ($d = \dim_k X$)

$$0 \to \mathcal{K}_n^{MW} \to \oplus_{x \in X^{(0)}} K_n^{MW}(k(x)) \xrightarrow{\partial} \oplus_{x \in X^{(1)}} K_{n-1}^{MW}(k(x); \det^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\partial} \dots$$
$$\xrightarrow{\partial} \oplus_{x \in X^{(q)}} K_{n-q}^{MW}(k(x); \det^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\partial} \dots$$
$$\xrightarrow{\partial} \oplus_{x \in X^{(d-1)}} K_{n-d+1}^{MW}(k(x); \det^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \xrightarrow{\partial} \oplus_{x \in X^{(d)}} K_{n-d}^{MW}(k(x); \det^{-1} \mathfrak{m}_x/\mathfrak{m}_x^2) \to 0$$

Looking at the terms in degree $n-1, n, n+1$, ones sees that an element x of $\tilde{\mathrm{CH}}^n(X)$ is represented by a finite formal sum

$$\sum_j q_j \cdot Z_j$$

where the Z_j are codimension n subvarieties of X, q_j is in $\mathrm{GW}(k(Z_j), \det \mathcal{N}_j)$, and $\mathcal{N}_j$ is the restriction to $\operatorname{Spec} k(Z_j)$ of the normal sheaf $(\mathcal{I}_{Z_j}/\mathcal{I}_{Z_j}^2)^\vee$. There is the coboundary condition $\partial(\sum_j q_j \cdot Z_j) = 0$, living in the twisted Witt groups of codimension one points of the Z_js, and all this is modulo the boundary of elements of the twisted K_1^{MW} of generic points of codimension $n-1$ subvarieties. One should think of these relations as a quadratic version of the divisor of rational functions.

Since $\langle u^2 v \rangle = \langle v \rangle$, we have canonical isomorphisms

$$\mathrm{CH}^n(X; \mathcal{L} \otimes \mathcal{M}^{\otimes 2}) \cong \mathrm{CH}^n(X; \mathcal{L})$$

For $f : Y \to X$ a proper map of smooth varieties of relative dimension d, and $\mathcal{L}$ an invertible sheaf on X we have the pushforward map

$$f_* : H^p(Y, \mathcal{K}_q^{MW}(\omega_f \otimes f^*\mathcal{L})) \to H^{p-d}(X, \mathcal{K}_{q-d}^{MW}(\mathcal{L}))$$

Here ω_f is the *relative dualizing sheaf* $\omega_f := \omega_{Y/k} \otimes f^*\omega_{X/k}^{-1}$, and $\omega_{Y/k} = \Omega_{Y/k}^{\dim Y}$ is the sheaf of top degree differential forms (similarly for $\omega_{X/k}$). This gives

$$f_* : \tilde{\mathrm{CH}}^n(Y, \omega_f \otimes f^*\mathcal{L}) \to \tilde{\mathrm{CH}}^{n-d}(X, \mathcal{L})$$

For a rank r vector bundle $p : V \to X$ with zero section $s_0 : X \to V$, we have

$$\omega_{s_0} = \det V$$

giving the pushforward

$$s_{0*} : \tilde{\mathrm{CH}}^m(X) \to \tilde{\mathrm{CH}}^{m+r}(V, p^* \det^{-1} V)$$

and the *Euler class*

$$e(V) := s^* s_{0*}(1_X) \in \tilde{\mathrm{CH}}^r(X, \det^{-1} V).$$

For $p_X : X \to \operatorname{Spec} k$ smooth and proper of dimension n we have the *quadratic degree*

$$\widetilde{\deg}_k := p_{X*} : \tilde{\mathrm{CH}}^n(X, \omega_{X/k}) \to \tilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k)$$

An *orientation* for a vector bundle $V \to X$ is an isomorphism $\rho : \det^{-1} V \xrightarrow{\sim} \omega_X \otimes \mathcal{L}^{\otimes 2}$ for some invertible sheaf $\mathcal{L}$. Given an orientation on a vector bundle V of rank $n = \dim_k X$, we have $\tilde{\deg}_k(e(V)) \in \mathrm{GW}(k)$ defined by applying the composition

$$\tilde{\mathrm{CH}}^n(X, \det^{-1} V) \xrightarrow{\rho_*} \tilde{\mathrm{CH}}^n(X, \omega_X \otimes \mathcal{L}^{\otimes 2}) \cong \tilde{\mathrm{CH}}^n(X, \omega_X) \xrightarrow{p_{X*}} \tilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k).$$

to $e(V)$.

The surjection $\mathcal{K}^{MW}_* \to \mathcal{K}^M_*$ extends to a surjection $\mathcal{K}^{MW}_*(\mathcal{L}) \to \mathcal{K}^M_*$, giving the map

$$\tilde{\mathrm{CH}}^n(X, \mathcal{L}) \to \mathrm{CH}^n(X)$$

In another direction, the isomorphisms $\mathcal{K}^{MW}_n(\mathcal{L}) \to \mathcal{W}(\mathcal{L})$ for $n < 0$ are compatible with multiplication by η, $\times\eta : \mathcal{K}^{MW}_n(\mathcal{L}) \to \mathcal{K}^{MW}_{n-1}(\mathcal{L})$, so extends to a map

$$\times\eta^N : \mathcal{K}^{MW}_n(\mathcal{L}) \to \mathcal{W}(\mathcal{L}), \quad N >> 0$$

giving the map

$$\tilde{\mathrm{CH}}^n(X, \mathcal{L}) \to H^n(X, \mathcal{W}(\mathcal{L}))$$

One a the functorialities for $H^n(X, \mathcal{W}(\mathcal{L}))$ similar to those for the twisted Chow-Witt groups, and the two comparison maps

$$\mathrm{CH}^n(X) \leftarrow \tilde{\mathrm{CH}}^n(X, \mathcal{L}) \to H^n(X, \mathcal{W}(\mathcal{L}))$$

are compatible with f^* and f_*. For the case of the degree maps, we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^n(X) & \longleftarrow & \tilde{\mathrm{CH}}^n(X, \omega_{X/k}) & \longrightarrow & H^n(X, \mathcal{W}(\omega_{X/k})) \\
\downarrow{\scriptstyle \deg_k} & & \downarrow{\scriptstyle \tilde{\deg}_k} & & \downarrow{\scriptstyle \bar{\deg}_k} \\
\mathbb{Z} & \xleftarrow{\text{rank}} & \mathrm{GW}(k) & \xrightarrow{\pi} & W(k)
\end{array}$$

for X smooth and proper of dimension n over k, with

$$\bar{\deg}_k = p_{X*} : H^n(X, \mathcal{W}(omega_{X/k})) \to H^0(\operatorname{Spec} k, \mathcal{W}) = W(k)$$

and with $\pi : \mathrm{GW}(k) \to W(k)$ the quotient map.

Noting that an element of $x \in \mathrm{GW}(k)$ is determined by $\mathrm{rank}(x) \in \mathbb{Z}$ and $\pi(x) \in W(k)$, it is often easier to work with the somewhat simpler Witt sheaf cohomology if one is mainly interested in "quadratic part" of enumerative invariants. Here are some examples.

Quadratic Bézout theorem The global part is very simple

Proposition 2.4. *Let $V \to X$ be a vector bundle of odd rank r. Then $e^{\mathcal{W}}(V) \in H^r(X, \mathcal{W}(\det^{-1} V))$ is zero.*

The Euler class is multiplicative with respect to direct sums (or exact sequences), so

$$e^{\mathcal{W}}(\oplus_j L_j) = 0$$

for line bundles L_j. However, for the quadratic Bézout theorem, one also needs the quadratic analog of the intersection multiplicities. This can be supplied by the Euler class with support and the purity theorem.

Let $V \to X$ be a rank r vector bundle, $s : X \to V$ a section and $Z \subset X$ a closed subset containing the locus $s = 0$. Then $e(V) := s^* s_{0*}(1_X) \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$ lifts canonically to the *Euler class with support* $e_Z(V, s) \in H^r_Z(X, \mathcal{K}_r^{MW}(\det^{-1} V))$.

The purity theorem is the following

Theorem 2.5. *Suppose $i : Z \to X$ is the inclusion of a smooth subvariety Z of a smooth variety X of codimension c, and let $\mathcal{L}$ be an invertible sheaf on X. Then the pushforward $i_* : H^{p-c}(Z, \mathcal{K}_{q-c}^{MW}(i^*\mathcal{L} \otimes \omega_i) \to H^p(X, \mathcal{K}_q^{MW}(\mathcal{L}))$ factors through an isomorphism*

$$i_* : H^{p-c}(Z, \mathcal{K}_{q-c}^{MW}(i^*\mathcal{L} \otimes \omega_i) \xrightarrow{\sim} H^p_Z(X, \mathcal{K}_q^{MW}(\mathcal{L}))$$

via the forget the support map $H^p_Z(X, \mathcal{K}_q^{MW}(\mathcal{L})) \to H^p(X, \mathcal{K}_q^{MW}(\mathcal{L}))$.

To apply this to Bézout's theorem, take our two curves C_1, C_2 defined by sections $F_i : \mathbb{P}^2 \to O_{\mathbb{P}^2}(d_i)$ and with $C_1 \cap C_2 = \{p_1, \ldots, p_r\}$. Let $Z = \{p_1, \ldots, p_r\}$. The section $s := (F_1, F_2)$ of $V := O_{\mathbb{P}^2}(d_1) \oplus O_{\mathbb{P}^2}(d_2)$ gives the Euler class with support

$$e_Z(V, s) \in H^2_Z(\mathbb{P}^2, \mathcal{K}_2^{MW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2)) \cong \oplus_j H^0(p_j, \mathcal{GW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2) \otimes \omega_{\mathbb{P}^2}^{-1}) \otimes k(p_j))$$

Now suppose that $-d_1 - d_2$ is odd, and recall that $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$. Then $\mathcal{GW}(\mathcal{O}_{\mathbb{P}^2}(-d_1 - d_2) \otimes \omega_{\mathbb{P}^2}^{-1}) \cong \mathcal{GW}$, and we have

$$e_Z(V, s) = \prod_j \tilde{m}(F_1, F_2, p_j) \in \oplus_j \mathrm{GW}(p_j)$$

defining the quadratic intersection multiplicity $\tilde{m}(s_1, s_2, p_j) \in \mathrm{GW}(p_j)$. Using the functoriality of pushforward, and the fact that the pushforward for $p_j \to \operatorname{Spec} k$ is the trace map $\mathrm{Tr}_{k(p_j)/k} : \mathrm{GW}(k(p_j)) \to \mathrm{GW}(k)$, we find

$$\tilde{\deg}_k(e(V)) = \sum_j \mathrm{Tr}_{k(p_j)/k}(\tilde{m}(F_1, F_2, p_j))$$

But since $e^{\mathcal{W}}(V) = 0$, this says that $\pi(\tilde{\deg}_k(e(V))) = 0$ in $W(k)$, that is, $\tilde{\deg}_k(e(V)) = m \cdot H$. Comparing with the classical Bézout theorem, we know that $m = d_1 d_2 / 2$, which is an integer, since exactly one of d_1, d_2 is even. This gives us the quadratic Bézout theorem.

Theorem 2.6. *Suppose we have plane curves $C_1, C_2 \subset \mathbb{P}^2_k$ of degree d_1, d_2, with no common components. Suppose in addition that $d_1 + d_2$ is odd Then*

$$\sum_j Tr_{k(p_j)/k}(\tilde{m}(F_1, F_2, p_j)) = \frac{d_1 d_2}{2} \cdot H$$

To round things out, it would be nice if we had a more explicit description of the quadratic intersection multiplicity. This is given by a quadratic refinement of the formula

$$m(C_1, C_2, p) = \dim_k \mathcal{O}_{\mathbb{P}^2, p}/(f_1, f_2)$$

where (f_1, f_2) are local defining equations for C_1, C_2 near an intersection point p.

For this, we need to make clear how our (canonical) isomorphism $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ gives rise to the isomorphism $\mathcal{GW}(\mathcal{O}_{\mathbb{P}^2}(-d_1-d_2)\otimes\omega_{\mathbb{P}^2}^{-1}) \cong \mathcal{GW}$.

The isomorphism $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ is given by choosing the global generator for $\omega_{\mathbb{P}^2}(3)$ to be the differential form

$$\Omega := X_0dX_1dX_2 - X_1dX_0dX_2 + X_2dX_0dX_1$$

so we have $\mathcal{O}_{\mathbb{P}^2}(-3) \cong \omega_{\mathbb{P}^2}$ by sending a local section λ of $\mathcal{O}_{\mathbb{P}^2}(-3)$ to to local section $\lambda\cdot\Omega$ of $\omega_{\mathbb{P}^2}$. This gives the isomorphism $\mathcal{O}_{\mathbb{P}^2}(-d_1-d_2+3) \cong \omega_{\mathbb{P}^2}^{-1}(-d_1-d_2)$ similarly. Writing $-d_1-d_2+3 = 2m$, we have the isomorphism

$$\phi : \mathcal{O}_{\mathbb{P}^2}(m)^{\otimes 2} \xrightarrow{\sim} \omega_{\mathbb{P}^2}^{-1}(-d_1-d_2),$$

and a *distinguished local section* of $\omega_{\mathbb{P}^2}^{-1}(-d_1-d_2)$ is a section of form $\phi(\lambda^2)$ for λ a local section of $\mathcal{O}_{\mathbb{P}^2}(m)$.

Take $p = p_j$ for some j and let $L = L(X_0, X_1, X_2)$ be a linear form with $L(p) \neq 0$. Choose local parameters t_1, t_2 generating $\mathfrak{m}_p \subset \mathcal{O}_{\mathbb{P}^2,p}$ such that

$$(L^{d_1+d_2}\cdot dt_1 \wedge dt_2)^{-1}$$

is a distinguished local section of $\omega_{\mathbb{P}^2}^{-1}(-d_1-d_2)$ and let $f_i = F_i/L^{d_i} \in \mathfrak{m}_p$. Choose $a_{ij} \in \mathcal{O}_{\mathbb{P}^2,p}$ so that

$$f_i = a_{i1}t_1 + a_{i2}t_2$$

and let e be the image of $\det(a_{ij})$ in $J := \mathcal{O}_{\mathbb{P}^2,p}/(f_1, f_2)$. J is a Artin local ring with residue field $k(p)$, so the surjection $J \to k(p)$ admits a (non-unique) splitting, making J a finite dimensional $k(p)$-algebra.

Proposition 2.7 (Scheja-Storch, Kass-Wickelgren)**.** *1. e is independent of the choice of the a_{ij} and generates the socle of J as $k(p)$-vector space.*
2. Let $\ell : J \to k(p)$ be a $k(p)$-linear form with $\ell(e) = 1$. Then $\tilde{m}(F_1, F_2, p) \in \mathrm{GW}(k(p))$ is represented by the quadratic form

$$q_{SS}(x) := \ell(x^2)$$

Example 2.8. The simplest case is when C_1 and C_2 intersect transversely at p and p is a k-point, so $J = k$. In this case, the image of a_{ij} in J is just $(\partial f_i/\partial t_j)(p)$, so e is the determinant of the Jacobian matrix $(\partial f_i/\partial t_j)(p)$, and q_{SS} is the rank one form $\langle 1/e\rangle \sim \langle e\rangle$.

Exercise Assume that at p, using coordinates (x, y) and a certain L gives a distinguished local section of $\omega_{\mathbb{P}^2}^{-1}(-d_1-d_2)$ at p, and that $f_i = F_i/L^{d_i}$. Compute the quadratic intersection multiplicity at $p = (0, 0) \in \operatorname{Spec} k[x, y]$ for the given (f_1, f_2)

a. $(f_1, f_2) = (x, 3y)$
b. $(f_1, f_2) = (x, y^2)$
c. $(f_1, f_2) = (y - x^2, y^2 - x^3)$
d. $(f_1, f_2) = (yx^2, y^2 - x^3)$.

Lines on a hypersurface

As for the Chow group, one can compute the quadratic count of th enumber of lines on a hyupersurface $X \subset \mathbb{P}^n$ of appropriate degree d by computing the degree of the Euler class of $\mathrm{Sym}^d(E_2^\vee)$, where $E_2 \to \mathrm{Gr}(2, n+1)$ is the tautological rank 2 subbundle of the trivial rank $n+1$ bundle. Since $\dim_k \mathrm{Gr}(2, n+1) = 2n-2$ and

$\mathrm{Sym}^d(E_2^\vee)$ has rank $d+1$, the condition on d is $d = 2n-3$. In this case $\mathrm{Sym}^d(E_2^\vee)$ has even rank $2n$, so one has the possibility of a non-zero Euler class. We need to check the orientation condition.

One has the Euler sequence for $\mathrm{Gr}(2, n+1)$:

$$0 \to E_2 \otimes E_2^\vee \to \mathcal{O}_{\mathrm{Gr}(2,n+1)}^{n+1} \otimes E_2^\vee \to T_{\mathrm{Gr}(2,n+1)} \to 0$$

$\det E_2^\vee = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(1)$ with respect to the Plücker embedding, and $\det E_2 \otimes E_2^\vee$ is trivial, so we have

$$\det T_{\mathrm{Gr}(2,n+1)} = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(n+1),\ \omega_{\mathrm{Gr}(2,n+1)} = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(-n-1)$$

We can compute $\det \mathrm{Sym}^d(E_2^\vee)$ by using the splitting principle again: If $E_2^\vee = M_1 \oplus M_2$, then

$$\mathrm{Sym}^d(E_2^\vee) = \oplus_{i=0}^d M_1^{\otimes d-i} \otimes M_2^{\otimes i}$$

so

$$\det \mathrm{Sym}^d(E_2^\vee) = (M_1 \otimes M_2)^{\sum_{i=1}^d i} = \mathcal{O}_{\mathrm{Gr}(2,n+1)}(\frac{d(d+1)}{2})$$

Since $d = 2n-1$, this is $\mathcal{O}_{\mathrm{Gr}(2,n+1)}((2n-3)(n-1))$ and so

$$\det{}^{-1} \mathrm{Sym}^d(E_2^\vee) \cong \omega_{\mathrm{Gr}(2,n+1)} \otimes \mathcal{O}_{\mathrm{Gr}(2,n+1)}((n-1)^2+1)^{\otimes 2}$$

which gives the orientation condition. We thus have

$$\begin{aligned} e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee)) &\in H^{2n-2}(\mathrm{Gr}(2,n+1), \mathcal{W}(\det{}^{-1} \mathrm{Sym}^d(E_2^\vee))) \\ &\cong H^{2n-2}(\mathrm{Gr}(2,n+1), \mathcal{W}(\omega_{\mathrm{Gr}(2,n+1)})) \end{aligned}$$

so we have

$$\tilde{\deg}_k(e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee))) \in W(k).$$

To compute this, we use the following general result

Theorem 2.9. *Let $V \to X$ be a rank 2 vector bundle. Then for d odd*

$$e^{\mathcal{W}}(\mathrm{Sym}^d V) = d!! e(V)^{d+1/2} \in H^{d+1}(X, \mathcal{W}(\det{}^{-1} \mathrm{Sym}^d V))$$

Here $d!! = d \cdot (d-2) \cdots 3 \cdot 1$.

In our case, we have

$$e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee)) = d!! e^{\mathcal{W}}(E_2^\vee)^{n-1} \in H^{2n-2}(\mathrm{Gr}(2,n+1), \mathcal{W}(\mathcal{O}_{\mathrm{Gr}(2,n+1)}(n-1)))$$

Wendt has computed the intersection ring of $H^*(\mathrm{Gr}(2,n+1), \mathcal{W}(*))$ and shows that

$$\tilde{\deg}_k(e^{\mathcal{W}}(E_2^\vee)^{n-1}) = \langle 1 \rangle \in W(k)$$

so

$$\tilde{\deg}_k(e^{\mathcal{W}}(\mathrm{Sym}^d(E_2^\vee))) = d!! \cdot \langle 1 \rangle \in W(k).$$

If we let $N_1(n) = \deg_k(c_{2n-2}(\mathrm{Sym}^{2n-3}(E_2^\vee))) \in \mathbb{Z}$, then we have the full quadratic degree

$$\tilde{\deg}_k(e^{CW}(\mathrm{Sym}^d(E_2^\vee))) = d!! \cdot \langle 1 \rangle + \frac{N_1(n) - d!!}{2} \cdot H \in \mathrm{GW}(k)$$

For the case of the cubic surface in $\mathbb{P}^3$, we have

$$\tilde{\deg}_k(e^{CW}(\mathrm{Sym}^3(E_2^\vee))) = 3 \cdot \langle 1 \rangle + 12 \cdot H \in \mathrm{GW}(k)$$

This recovers the first such computation, by Kass-Wickelgren, who used a more explicit computation of the Euler class via the quadratic local multiplicities.

Quadratic Gauß-Bonnet and the quadratic Riemann-Hurwitz formula

Theorem 2.10. *Let X be smooth and proper over a field k. Then*

$$\chi(X/k) = \tilde{\deg}_k(e^{CW}(T_{X/k})) \in \mathrm{GW}(k)$$

and the image $\pi(\chi(X/k))$ of $\chi(X/k)$ in $W(k)$ is given by

$$\pi(\chi(X/k)) = \tilde{\deg}_k(e^{\mathcal{W}}(T_{X/k})) \in W(k)$$

Note: This says in particular that $\chi(X/k) = m \cdot H$ for some integer m if $\dim_k X$ is odd.

We will say a bit about the proof in Lecture 3. A consequence is a quadratic version of the Riemann-Hurwitz formula

Theorem 2.11. *Let $f : X \to C$ be a morphism of a smooth proper k-scheme X of dimension n to a smooth projective curve C. Suppose that the induced section $df : \mathcal{O}_X \to \Omega_X \otimes f^*\omega_C^{-1}$ has isolated zeros $p_1, \ldots, p_r$ with quadratic multiplicities $\tilde{m}_i \in W(k(p_i))$. If n is odd, we suppose in addition that $\omega_C \cong \mathcal{L}^{\otimes 2}$ for some invertible sheaf on C. Then*

$$\pi(\chi(X/k)) = \sum_i Tr_{k(p_i)/k}\tilde{m}_i \in W(k).$$

Since $\det(\Omega_X \otimes f^*\omega_C^{-1}) = \omega_X \otimes f^*\omega_C^{-n}$, our assumption that $\omega_C \cong \mathcal{L}^{\otimes 2}$ if n is odd says that we have the orientation condition needed to define the local quadratic multiplicities

$$\tilde{m}_i := e^{\mathcal{W}}_{p_i}(\Omega_X \otimes f^*\omega_C^{-1}, df) \in H^n_{p_i}(X, \mathcal{W}(\omega_X \otimes f^*\omega_C^{-n})) \cong H^n_{p_i}(X, \mathcal{W}(\omega_X)) \cong W(k(p_i))$$

The proof follows the same idea as for the classical case: one computes $\tilde{\deg}_k e^{\mathcal{W}}(\Omega_{X/k} \otimes f^*\omega_{C/k}^{-1})$ as $\sum_i \mathrm{Tr}_{k(p_i)/k}\tilde{m}_i$ and then uses

Proposition 2.12. *Let V be a rank r vector bundle on a smooth k-scheme X and let L be a line bundle on X. If r is odd, we suppose that $L \cong M^{\otimes 2}$ for some line bundle M. Then*

$$e^{\mathcal{W}}(V \otimes L) = e^{\mathcal{W}}(V) \in H^{2r}(X, \mathcal{W}(\det^{-1} V)) \cong H^{2r}(X, \mathcal{W}(\det^{-1}(V \otimes L)))$$

One also has an explicit formula for the $\tilde{m}_i$ using the quadratic form on the local Jacobian rings

$$J(df)_{p_i} = \mathcal{O}_{X,p_i}/(\ldots, \partial f/\partial t_i, \ldots)$$

with respect to suitably chosen coordinates $t_1, \ldots, t_n$ at p_i. In fact, take $p = p_i$ a point with $df = 0$. Let $q = f(p)$ and let $t \in \mathfrak{m}_q \subset \mathcal{O}_{C,q}$ be a local parameter. Let $x_1, \ldots, x_n \in \mathfrak{m}_p \subset \mathcal{O}_{X,p}$ be local parameters. If n is odd, we let $\rho : \mathcal{L}^{\otimes 2} \to \omega_C$ be the chosen "orientation" and we assume that the local generator dt of $\omega_{C,q}$ is of the form $\rho(\lambda^2)$ for λ a local generator of $\mathcal{L}$ near q. Let $g = f^*(t) \in \mathfrak{m}_p$, giving the partial derivatives $\partial g/\det x_i$, $i = 1, \ldots, n$. Let $J(f,p) = \mathcal{O}_{X,p}/(\partial g/\det x_1, \ldots, \partial g/\det x_n)$ and choose elements $a_{ij} \in \mathcal{O}_{X,p}$ with

$$\partial g/\det x_i = \sum_{j=1}^n a_{ij} x_j$$

Let $e_{SS} \in J(f,p)$ be the image of $\det(a_{ij})$. The fact that df has an isolated zero at p implies that $J(f,p)$ is an Artin k-algebra, so contains the residue field $k(p)$. Let

$\ell : J(f,p) \to k(p)$ be a $k(p)$ linear map with $\ell(e_{SS}) = 1$ and define the quadratic form $q^{SS}_{f,p}$ on $J(f,p)$ with values in $k(p)$ by

$$q^{SS}_{f,p}(x) = \ell(x^2)$$

Then the local Euler class $\tilde{m}_i^{CW} := e^{CW}_{p_i}(\Omega_X \otimes f^*\omega_C^{-1}, df) \in \mathrm{GW}(k(p))$ is represented by $q^{SS}_{f,p}$.

Exercises

1. Suppose X and C are both smooth curves and $f : X \to C$ a finite cover. Take $p \in X$ and suppose we have local parameters x at p and t at $q := f(p)$ such that $f^*(t) = ux^n$ for $u \in \mathcal{O}^\times_{X,p}$ a unit. Suppose that n is prime to the characteristic and that dt satisfies the appropriate orientation condition. Compute the quadratic multiplicity $e^{CW}_{p_i}(\Omega_X \otimes f^*\omega_C^{-1}, df) \in \mathrm{GW}(k(p))$.

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