Formalization of Cohomology Theories

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1 Overview of the Field

In the last few years, formalized mathematics libraries, such as Lean's `mathlib` [The20], Coq's *Mathematical Components* [MT17] and Isabelle's *Archive of Formal Proofs*, have made significant progress in expanding their coverage, and now include a substantial portion of an undergraduate and graduate-level mathematics curriculum. The advances in computer formalization of relatively elementary mathematics means that formalization of definitions and theorems from *research-level mathematics* are now within reach. However, at this time, none of these libraries have a sufficiently robust way to handle *(co)homology*, which is an invaluable tool in several branches of modern mathematics. In fact, (co)homology theories can take diverse forms and it is far from obvious how to treat these uniformly in an interactive proof assistant such as the ones mentioned above.

Unlike traditional informal mathematics, much of the difficulty of formalization lies in carefully setting up definitions, and designing a sufficiently thorough interface so that these definitions can be used without any headaches. This is fundamentally different from formalizing long proofs about elementary objects, and it requires a level of creativity that goes beyond simply copying a textbook mathematical definition. Investigating the pain points in a formalization attempt can also bring light to subtleties and emphasize a fresh point of view to existing practice. A prominent example is the pervasive use of filters, already introduced by Bourbaki, as a unifying concept for all the various notions of limits and convergence in analysis/topology. Another example is the synthetic approach to algebraic topology coming from Homotopy Type Theory, which originated from fundamental research in computer verification of mathematics.

One long-term goal of formalized mathematics is to provide a tool usable for mathematics researchers. Formalizing areas of mathematics relevant to active modern research is essential to achieving this goal. As we see it, developing an interface for the various flavors of (co)homology is a natural next step that will help formalized mathematics become a serious tool in mathematics research, and this is what the proposed workshop will focus on.

This workshop brought together mathematicians with some formalization experience with formalization experts of various systems including Coq, Agda, and Lean, to discuss the existing progress in formalizing (co)homology and to collaborate on further development of these areas. The libraries of formalized mathematics mentioned above are developing at an extremely rapid pace. Several basic forms of (co)homology and/or related constructions already exist in some of these libraries, as we will highlight in the following sections.

2 Recent Developments and Open Objectives

2.1 Singular and simplicial homology

Harrison [Har18] has defined singular homology in the HOL Light prover [Har96], pursuing the theory as far as the verification of the Eilenberg-Steenrod axioms and consequences such as the Brouwer fixed-point theorem and the Jordan-Brouwer separation theorem. This work has also been ported[1] to Isabelle/HOL [NWP02]. Heras, Coquand, Mörtberg and Siles [HCMS13] have formalized a synthetic approach to simplicial and persistent homology in Coq, as far as we know this last development is not connected to existing topology libraries.

In Lean, Brendan Murphy has formalized singular homology and shown it obeys the Eilenberg-Steenrod axioms en route to a proof of the Brouwer fixed-point theorem[2]. Joël Riou has formalized the Dold-Kan correspondence between simplicial objects and non-negatively-graded chain complexes for an arbitrary preadditive category[3]. Both of these introduce essential tools for further mechanization of algebraic topology.

An immediate objective in this direction is in the further development of the homotopy theory of topological spaces and simplicial sets integrating and extending these tools.

Objective. Define simplicial homotopies, prove that they induce a chain-homotopy of chain complexes. Prove that a topological homotopy induces a simplicial homotopy on singular simplicial sets. Prove that the singular simplicial set of the geometric realization of a simplicial set X is homotopic to X.

The Whitney embedding theorem has recently been formalized in Lean[4], and many of the ingredients necessary for this theorem have been formalized in Isabelle. A potential application of this development, related to singular (co)homology, which is now within reach is in proving the existence of triangulations of smooth manifolds.

Objective. Define abstract simplicial complexes, construct the associated simplicial set to a simplicial complex, and prove that the homology of the associated simplicial set is equal to the combinatorial homology of the original complex. Prove that every smooth compact manifold admits a triangulation, as an abstract simplicial complex.

2.2 Sheaf cohomology

The notion of a sheaf on a site has been recently formalized in Lean, while some other systems have a definition of a sheaf on a topological space. However, sheaf cohomology has yet to be constructed, although the necessary ingredients (derived functors[5], projective resolutions[6], etc.) for this definition have been recently formalized in Lean. An initial objective in this direction is to build on the formalization of homological algebra in an abstract abelian category.

Objective. Formalize the notion of a δ-functor. Prove that a short exact sequence in an abelian category induces a long exact sequence of derived functors. Prove that derived functors are universal δ-functors. It's worthwhile to note that the notion of a δ-functor has been formalized as part of the of the liquid tensor experiment. However, the supporting theory around this definition has yet to be developed.

One particularly exciting objective in this context is to relate sheaf cohomology, constructed in the general setting described above, to the classical topological context. For example, Lean's `mathlib` currently possesses a definition of a topological vector bundle, while the definition in the smooth and analytic categories

[1] https://isabelle.in.tum.de/dist/library/HOL/HOL-Homology/index.html
[2] https://github.com/Shamrock-Frost/BrouwerFixedPoint
[3] https://github.com/leanprover-community/mathlib4/blob/master/Mathlib/AlgebraicTopology/DoldKan/Equivalence.lean
[4] https://leanprover-community.github.io/mathlib_docs/geometry/manifold/whitney_embedding.html
[5] https://leanprover-community.github.io/mathlib_docs/category_theory/derived.html
[6] https://leanprover-community.github.io/mathlib_docs/category_theory/preadditive/projective_resolution.html

should be fairly accessible as well. Relating (continuous/smooth/analytic) line bundles to sheaf cohomology would be an extremely interesting project which is nevertheless quite tractable with existing technology.

Objective. Prove that the singular cohomology of a topological space X (satisfying suitable axioms) with constant coefficients agrees with the sheaf cohomology of the associated locally constant sheaf. Relate the first sheaf cohomology to the classification of line bundles.

2.3 Cohomology in algebraic and arithmetic geometry

Étale cohomology is amongst the most powerful concepts introduced in arithmetic geometry in the 20th century. Particularly in the form of ℓ-adic cohomology, it shares many properties with singular cohomology of complex algebraic varieties. Recently, Bhatt and Scholze have rephrased ℓ-adic cohomology in terms of the sheaf cohomology of the proétale site on a scheme, leading to a conceptually clearer definition which is more amenable to formalization.

Another approach toward formalization, which works in the case of a field k, arises from the fact that the étale cohomology of k is canonically isomorphic to the group cohomology of the absolute Galois group of k, which admits a concrete description in terms of (in)homogeneous continuous cocycles and coboundaries. This specialisation goes under the name of *Galois cohomology*, and it plays an important role in number theory.

The preliminary ingredients for these cohomology theories have now all been formalized in Lean's `mathlib`: schemes[7], Galois groups with the Krull topology[8], Grothendieck topologies[9]. Furthermore, Amelia Livingston has recently formalized group cohomology in Lean[10] setting the stage for Galois cohomology.

Objective. Formalize a general notion of proétale cohomology, and specialize to ℓ-adic cohomology and/or Galois cohomology. Construct comparison isomorphisms between the étale cohomology of a field and the (continuous) group cohomology of its absolute Galois group.

Another fundamental construction in algebraic geometry, which is defined in terms of intersection theory, is the *Chow group* of codimension i algebraic cycles on an algebraic variety. These can be related to singular and/or étale cohomology via regulator maps, and the relationship between the two theories is a central theme in modern arithmetic and algebraic geometry. A long term goal in the formalization of arithmetic/algebraic geometry is to build toward a formalization of algebraic cycles, rational equivalence, and Chow groups, as well as their relationship with other cohomology theories. However, the current state of arithmetic/algebraic geometry in formal mathematics libraries is not sufficiently developed for this to be a reasonable goal by 2023. Nevertheless, we believe that a formal construction of the Chow group of codimension 1 cycles and its relationship with (algebraic) line bundles is within reach.

Objective. Formalize the definition of the Chow group of codimension 1 cycles up to rational equivalence, the definition of the Picard group, and relate the two.

2.4 Derived categories, cohomological functors, and representability

One primary motivation for the introduction of derived categories was because they can act as a repository to house all cohomological functors, $\mathrm{Ext}^j(\mathcal{E}, \mathcal{F}), \mathrm{Tor}_j(M, N), H^*(X, \mathcal{F})$ and others. As such, a robust framework for derived categories is an important tool for formalizing results about general cohomological functors. Joël Riou has recently formalized the definition of derived and triangulated categories, and has started developing the surrounding theory.

[7] `https://github.com/leanprover-community/mathlib4/blob/master/Mathlib/AlgebraicGeometry/Scheme.lean`

[8] `https://github.com/leanprover-community/mathlib4/blob/master/Mathlib/FieldTheory/KrullTopology.lean`

[9] `https://github.com/leanprover-community/mathlib4/blob/master/Mathlib/CategoryTheory/Sites/Grothendieck.lean`

[10] `https://github.com/leanprover-community/mathlib4/blob/master/Mathlib/RepresentationTheory/GroupCohomology/Basic.lean`

One particularly impressive tool is the *Brown Representability Theorem* which gives simple conditions for a cohomological functor $F : \mathcal{T} \to Ab$ to be representable in $\mathcal{T}$. Usually the category $\mathcal{T}$ must be endowed with additional structure in this context, e.g. $\mathcal{T}$ may be a triangulated category.

Objective. Formalize the statement and proof of the Brown Representability theorem.

Brown Representability provides a useful tool across different disciplines. Two particular applications are to the existence of Eilenberg-Maclane spaces in Algebraic Topology [Bro62] and to establishing Grothendieck duality in Algebraic Geometry [Nee96]. A formalized proof of Brown Representability will help spur development around Algebraic Topology (constructing the stable homotopy category) and around Algebraic Geometry (quasi-coherent sheaves).

3 Presentation Highlights

3.1 Monday

For the first day of the workshop, we searched for speakers to give "big-picture" talks which would lay out a roadmap to the formalization of some cohomology theory. In this way we hoped to stimulate informal work during the week, and to set the tone of the talks: informal, as much about work to come as work done already.

3.1.1 Iurii Kudriashov: The path to the formalization of de Rham cohomology

De Rham cohomology has been a longstanding goal for formalization, see e.g. [Gou22], and has partly shaped the direction of recent formalization work by a number of people: smooth manifolds, smooth vector bundles, the divergence theorem, a very general version of the change of variables formula.

In his talk, Kudriashov outlined the current status of these prerequisites and the steps which remain: defining continuous alternating maps and the smooth vector bundle of continuous alternating maps on a smooth manifold; defining the exterior derivative d locally and globally; proving that $d^2 = 0$. The whole definition seems very much within reach and progress was made at the workshop, see §4.

This is of course just what is needed for the bare definition of de Rham cohomology; there was also informal discussion of what would be needed for fundamental theorems about de Rham cohomlogy, such as the Poincaré lemma.

3.1.2 Emily Witt: Local cohomology and Macaulay2

Abstract: Introduced by Grothendieck in the 1960s, local cohomology is an essential tool in commutative algebra, and also capture geometric and topological properties. These modules, associated to an ideal of a Noetherian ring and module over this ring, are typically very large (e.g., not finitely generated), making it difficult to access this data. We touch on some steps needed to formalize local cohomology, while also giving a brief introduction to the open-source computer algebra system Macaulay2.

Witt, an algebraic geometer, is an expert on the Macaulay2 computer algebra system and a relative newcomer to formalized mathematics. Her talk bridged these areas: an introduction to local cohomology, a demonstration of computing local cohomology in Macaulay2, and an outline of three possible approaches to defining local cohomology formally. Some of these were pursued during the workshop, see §4.

Direct interaction between computational algebra software and interactive theorem provers is still in its early days, but it is on many people's minds.

3.1.3 Oliver Nash: On the formalisation of topological K-theory

Abstract: After providing some motivation I will explore various approaches to the formalisation of topological K-theory using Lean's Mathlib library. In fact we shall see that we can get quite far, though of course some gaps in the library will be encountered. As a result I will assemble a list of attractive formalisation targets, some easy, some not-so-easy.

Oliver Nash spoke on topological K-theory. His delightful talk proposed several definitions of K-groups and higher K-groups, each of which are nearly but not quite ready to be formalised; work went on during the week to fill in the various holes which he had highlighted, and we are confident that soon a basic K-theory development will join mathlib. As mentioned above, it's not good enough to just give a definition which is strictly speaking mathematically correct; one also needs to show that the definition is actually usable within the system. Reid Barton suggested that a good test theorem for this would be Bott periodicity; Nash has already begun to think about this. See §4 for comments about progress made in this area during the workshop.

3.2 Tuesday

The bare definition of singular cohomology has existed in Lean for some time, but substantial further work is require to build the supporting theory around it. On Tuesday morning there were two talks about singular homology and cohomology which discussed work in this direction.

3.2.1 Scott Morrison: Künneth formulas

This talk explored the boundary of what is currently possible around singular cohomology, focusing mostly on Künneth formulas. Morrison's talk walked through an outlined Lean write-up with most proofs not filled in yet, which was made available to the participants at the start of the lecture.[11] This file contained several formalization "targets" acting as a skeleton for the formalization of Künneth formulas. These targets included:

- The homology of $\mathbb{RP}^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$.
- A statement of the Künneth formula for products of topological spaces.
- The monoidal structures on graded objects, complexes, the homotopy category of complexes, etc.
- The assertion that several functors are (lax) monoidal, including the singular simplicial set functor from topological spaces to simplicial sets, the singular chain complex functor, the graded homology functor, etc.
- An explanation of how one can obtain a general Künneth formula using the various monoidal structures and functors mentioned above.

3.2.2 Brendan Murphy: Formalizing the Brouwer Fixed Point Theorem in Lean

Abstract: This talk discusses the results and methodology of a project to formalize the Brouwer Fixed Point theorem in Lean. There was no prior formalization of a (co)homology theory in Lean, so we decided to develop the basic theory of singular homology (i.e. prove the Eilenberg-Steenrod axioms for it). Notably this formalization uses the acyclic models theorem to prove homotopy invariance of singular homology; we discuss this and other design choices. This project depends on mathlib and on (the homological algebra developed for) the Liquid Tensor Experiment.

Murphy reported on his impressive work formalizing the Brouwer Fixed Point Theorem using Lean3, as well as several important prerequisites including the calculation of the homology of spheres, and a verification of the Eilenberg-Steenrod axioms. Murphy's work shows that the bare definition of singular homology that has existed for some time is actually useable in practice.

3.2.3 Matthieu Piquerez: Formalization of diagram chasing as a first-order logic in Coq (remote)

Abstract: Diagram chasing is at the heart of many powerful tools in mathematics, for instance spectral sequences. Unfortunately, their usage requires a lot of tedious and technical calculations. For instance, one has to check the commutativity of many diagrams. These technicalities are often not detailed in papers, and can be a source of mistakes. This motivates the development of a formalized library to do diagram chasing on

[11] https://github.com/leanprover-community/mathlib/tree/tuesday_9am

computer. In particular, a large part of the above mentioned computations can be automatized. In this talk, I will present the key points of such a library I am developing with Assia Mahboubi in Coq. In particular, I will explain that all the diagram chasing statements can be restated in a very simple language (in a first order logic), and I will state a formalized version of the often used duality meta-theorem. I hope that this library will be the first step of the development of a powerful library of tools in homological algebra.

Piquerez described his joint work with A. Mahboubi on the development of a domain specific language, within Coq, for diagrams in categories and a decision procedure for the commuitativity of such diagrams (under certain assumptions). He also described how several categorical properties can be rephrased as the commutativity of a class of diagrams.

3.2.4 María Inés de Frutos Fernández: Cohomology in Number Theory

Abstract: I will discuss recent and ongoing work in two relatively long-term number theory formalization projects. The first one, joint with Filippo Nuccio, consists on formalizing the theory of local fields and the main statements and proofs of Local Class Field Theory, following the cohomological approach. The second one consists on formalizing the definition and basic properties of Fontaine's period rings, in particular B_{HT}, B_{dR} and B_{crys}, as a starting point towards a formalization of p-adic Hodge theory. Part of this formalization is joint work with Antoine Chambert-Loir.

De Frutos Fernández discussed her ongoing work on the formalization of p-adic period rings using Lean, which is partly joint work with Chambert-Loir. A consequence of this work is that it will soon be possible to state the comparison theorems from p-adic Hodge theory. The organizers find it quite remarkable that complicated objects appearing in number theory, such as these p-adic period rings, are now amenable for formalization.

3.3 Wednesday

On Wednesday we had several talks around the theme of profinite and condensed mathematics.

3.3.1 Sam van Gool: Frames, profinite structures and sheaves

Abstract: This talk will be about an algebraic way of looking at topological, profinite structures. While it will be mostly a "traditional pen and paper" talk, I hope that, in the context of this workshop, it will provide an impulse to think together about how these things might be formalized, and about how they might fit with previous work done already in Mathlib and the LTE.

Van Gool's engaging talk connected concepts from topology, order theory, logic, computer science, and formal mathematics. Progress on the formalization of some of these topics was made during the workshop; see §4. Van Gool also provided an extended abstract which we include below.

> Frames give an algebraic, but infinitary, way of constructing and studying topological spaces. For many of the spaces occurring in practice, it is possible to replace frames by bounded distributive lattices, which have the advantage of being finitary. Formally, what one gets is a dual equivalence ("Stone duality") between such lattices and a category of spectral spaces. A theorem of Hochster shows that these are exactly the spaces arising as Zariski spectra of rings, explaining why they also appear naturally in geometry.
>
> Now, *Hausdorff* spectral spaces are the same thing as profinite sets, and Stone duality restricts to a well-known duality between profinite sets and Boolean algebras. It's a somewhat less well-known fact that, outside the Hausdorff setting, spectral spaces are exactly profinite T0-spaces, giving a convenient abstract viewpoint on Stone duality. This talk explained how profinite T0-spaces can be alternatively and beneficially understood as profinite posets, also known as Priestley spaces. It ended with an indication of some of the speaker's own research (joint with Mai Gehrke) regarding what sheaves on such spaces look like when taken through the Stone duality.[12]

[12] https://github.com/leanprover-community/mathlib4/pull/4593

3.3.2 Dagur Ásgeirsson: Discrete Condensed Sets

Abstract: There are several reasonable ways to give a "discrete" condensed structure to a set, modelling the behaviour of discrete topological spaces. In this talk I will present a formalisation in Lean of three of those, each useful in its own way, and isomorphisms between them. I will recall the few prerequisites from condensed mathematics before telling the story of the formalisation process. This work builds on definitions and theorems from the Liquid Tensor Experiment, and by extension the Lean mathematical library mathlib.

Ásgeirsson's talk explained a number of ways to define the notion of a *discrete condensed set*, and how these approaches are naturally equivalent. He has formalized these definitions and theorems in Lean, building on work done in the liquid tensor experiment. Thus far, this work is part of Ásgeirsson's project on the formalization of *solid condensed abelian groups*.[13]

3.3.3 Joël Riou: Formalization of derived categories in Lean (remote)

Abstract: Since they were introduced in the 1960s by Grothendieck and Verdier, derived categories of abelian categories have played a crucial role in the formulation of cohomological statements and their proofs. In this talk, I shall discuss a formalization of derived categories in Lean/mathlib (initially in Lean 3, but a significant part has already been ported to Lean 4). This project includes a refactor of the definition of the homology of complexes, the verification of the axioms of triangulated categories for the homotopy category of cochain complexes in an additive category, and a localization theorem for triangulated categories. Some of these ingredients were already obtained in the Liquid Tensor Experiment, but the newest part is the localization theorem which allows the definition of the derived category as the localization of the homotopy category of cochain complexes with respect to quasi-isomorphisms. I have also obtained various basic results involving derived categories, e.g. the full embedding of an abelian category in its derived category. (The Lean 4 code is available in my mathlib4 branch `jriou_localization`.)

Riou spoke about the formalization of homological algebra, including several lessons learned while working on the liquid tensor experiment. He presented his extensive redesign and expansion of `mathlib`'s homological algebra library.

3.4 Thursday

On Thursday all the talks were about homotopy theory.

3.4.1 Floris van Doorn: What can we learn from formalizations in homotopy type theory?

Abstract: Only the very basics of homotopy theory have been formalized in proof assistants in the usual way. However, using a special kind of type theory - called homotopy type theory , or HoTT for short - one can do various arguments in homotopy theory in a purely synthetic way. The types of the type theory are themselves considered to be (nice) topological spaces, and the equality type corresponds to paths in this space. HoTT provides very convenient concepts such as path induction, univalence and higher inductive types. Using these techniques many important theorems of homotopy theory have been proved and even formalized in HoTT, such as Blaker's-Massey theorem and the Serre spectral sequence.

Van Doorn's talk gave an excellent introduction to homotopy theory theory and what has been formalized thus far in synthetic homotopy theory.

3.4.2 Reid Barton: Simplicial homotopy theory

Barton's talk focused on the Quillen adjunction between topological spaces and simplicial sets. He explained which ingredients of this theorem have already been formalized, and what is left to do. A crucial part of the proof involves an intricate (transfinite) induction step which is commonly called the "small object argument".

[13] `https://github.com/dagurtomas/lean-solid`

It turns out that even *stating* the assertion of the small object argument precisely is difficult on paper, let alone in an ITP. Barton gave a precise explanation of how this part of the proof could be formalized.

3.4.3 Axel Ljungström: Cohomology Theory and Brunerie Numbers in Cubical Agda (remote)

Abstract: In his 2016 PhD thesis, Guillaume Brunerie gave a constructive definition, in Homotopy Type Theory, of a number n satisfying $\pi_4(S^3) = \mathbb{Z}/n\mathbb{Z}$. This number, nowadays called the 'Brunerie number', quickly rose to infamy in the type theory community: despite being constructively defined, it is not at all obvious that the Brunerie number is actually ± 2. In fact, with the advent of constructive proof assistants such as Cubical Agda, we should not have to figure this out by ourselves—Agda should be able to compute/normalise n for us. Unfortunately, also this fails: Agda runs out of memory. For this reason, Brunerie had to 'manually' show that $n = \pm 2$. This talk is loosely based on the recent formalisation of this 'manual' calculation of the Brunerie number. Since the key component in this calculation is cohomology theory, I will give you a brief introduction to the definition of cohomology in HoTT. I will also discuss which related constructions and theorems we have (and have not yet) formalised. On a more practical note, I will show you how new 'Brunerie numbers' appear everywhere in the computation of cohomology rings of spaces and try to convince you that, from this point of view, a fast constructive/fully computational proof assistant could potentially save you hours of work when formalising cohomology computations.

Ljungström explained how to rewrite the (well-known) proof that $\pi_4(S^3)$ is isomorphic to $\mathbb{Z}/2$ in such a way that the order of $\pi_4(S^3)$ can be calculated by the computer directly, using cubical Agda. He discussed the various optimizations and modifications that made such a calculation possible.

3.5 Friday

3.5.1 Amelia Livingston: Group cohomology in Lean

Abstract: Mathlib now has a definition of group cohomology, and a proof it agrees with some Ext groups. I will talk about the code this involved, which used the existing simplicial objects library. I will also describe the group cohomological results I have formalised but which are not yet in Mathlib. Finally, I will discuss the rest of the group cohomology to-do list, much of which will require porting material from the Liquid Tensor Experiment.

Livingston discussed her work on the formalization of group cohomology and the related theory. Her work relates an explicit definition of group cohomolgy in terms of cochains to the more abstract definition in terms of ext groups, and illustrated how this relationship can be used to do explicit calculations. Her work also includes the formalization of the long exact sequence associated to a short exact sequence of modules, and the inflation-restriction sequence.

3.5.2 Bohua Zhan: Formalizations in set theory: challenges and opportunities (remote)

Abstract: Set theory is widely considered to be the "standard" foundation of mathematics. However, the use of set theory to formalize mathematics is rare today. Some possible reasons include that lack of type checking leads to many nonsensical expressions, and many deductions that could be performed by type inference need to be done by hand. However, it may be possible to address such difficulties using appropriate abstractions and proof automation, after which the advantages of set theory will become more significant. I will discuss my past experiences with such issues, and speculate what could be done to make set theory more viable for formalized mathematics in the future.

The final talk of the workshop was delivered by Zhan, who spoke about the design of a new proof assistant, called `HolPy`[14], which incorporates ideas from both type theory and set theory.

[14] `https://github.com/bzhan/holpy`

4 Scientific Progress Made

The meeting brought together mathematicians from a broad range of specializations, ranging from formalization experts to those merely interested. The participants were encouraged to form groups with a dynamically varying composition. Apart from working together to teach and practice formalization skills, the participants produced an impressive amount of newly formalized material, to be contributed to the Lean mathematical library. Particular highlights include:

- Heather Macbeth gave a formal definition of Riemannian manifolds.[15]
- Van Gool, Mayer, Murphy and Baanen contributed a formalization of the adjunction between lattices and topological spaces which appeared in the talk of Van Gool.[16]
- Morrison contributed a definition of presheaves of modules over presheaves of rings.[17]
- Macbeth, Van Doorn and Doll fixed a small error in mathlib's formal definition of a vector bundle.
- Peter Nelson who was a participant of the workshop had been formalizing matroid theory in Lean for a number of years. His definition of a "matroid" changed as a consequence of discussions with Alex Best during the workshop. This work is now being added to mathlib.
- Heather Macbeth, Johan Commelin and Adam Topaz worked on formalizing the sheaf of holomorphic functions on a complex manifold.[18]
- Iurii Kudriashov contributed a formalization of continuous alternating linear map, which is a key ingredient necessary for the construction of de Rham cohomology, as outlined in his talk.[19]
- Scott Morrison and Emily Witt contributed a definition of local cohomology to mathlib, right after Witt's lecture.[20] Later in the week this contribution was expanded with the help of Jake Levinson and Sam van Gool.[21]
- Adam Topaz formalized the definition of K_0 after the talk of Oliver Nash. This is a key ingredient in the construction of K-theory.
- Some of the targets outlined in Morrison's talk on the Künneth formula were resolved by Macbeth and Best.
- Joël Riou refactored the definition of homology, following the strategy he outlined in his lecture.[22][23][24]
- Several participants of the meeting engaged in reviewing work by others, such as proposed contributions to mathlib. The workshop's collaborative environment helped facilitate some of their work.

Various additional discussions were had throughtout the week and additional partial progress was made that we did not list above. For example, Barton started discussing the formalization of the small object argument with Hazratpour and Murphy. Also, Ásgeirsson and Commelin started work on the formalization of Nöbeling's theorem which has recently been completed and contributed to mathlib by Ásgeirsson[25].

In addition, we have heard from several participants that presentations, discussions and collaborations during the workshop have afforded them progress and new collaborations in the work.

[15] https://github.com/leanprover-community/mathlib/tree/hrmacbeth-riemannian2
[16] https://github.com/leanprover-community/mathlib4/pull/4593
[17] https://github.com/leanprover-community/mathlib4/pull/4670
[18] https://github.com/leanprover-community/mathlib/pull/19094
[19] https://github.com/leanprover-community/mathlib4/pull/5678
[20] https://github.com/leanprover-community/mathlib/pull/19061
[21] https://github.com/leanprover-community/mathlib/pull/19105
[22] https://github.com/leanprover-community/mathlib4/pull/4197
[23] https://github.com/leanprover-community/mathlib4/pull/4204
[24] https://github.com/leanprover-community/mathlib4/pull/4388
[25] https://github.com/leanprover-community/mathlib4/pull/6286

Filippo Nuccio The workshop offered many occasions to discuss in detail different aspects of the current state-of-the-art of the formalization of cohomology theories in Lean. Exchanges with one of the speakers about her development of group cohomology helped figure our the next steps needed to formalize the statement, and eventually the proofs, of local class field theory in cohomological terms. I also had fruitful discussions with a mathlib maintainer regarding complete spaces, and her suggestions led to overcoming a difficulty that has been stopping me for quite some time in the formalization of local fields. Beyond these more specific examples, all the exchanges about Lean4 and the shape that mathlib4 is taking proved extremely interesting and useful.

Oliver Nash By attending this conference I learned just how many cohomology theories are already within range of computer formalisation. Furthermore I saw that these theories are ready, not just for proof-of-concept formalisation, but rather to be developed as fully-fledged theories in Lean's Mathlib library. As a differential geometer, large branches of my subject cannot be discussed without singular cohomology, de Rham cohomology, K-theory, and group cohomology and I was delighted to see each of these subjects discussed. As well as the talks, I hugely benefited from face-to-face interaction with many colleagues, especially those whose work is less closely connected to my own.

Alex J. Best The workshop also provided me with the opportunity to take a closer look and begin building on some recent developments in the field, such as in homological algebra itself, but also surrounding infrastructure such as interactive visualizations for diagram chases. A nontrivial portion of my time during the week was also spent helping other participants (both experts and those new to the area) with issues arising in their formalization work. I believe the ease of communication and collaboration here also helped accelerate progress in this area, and would have been much harder to accomplish without such a collaborative in person meeting. It was also very helpful to have lengthy informal discussions with other participants concerning useful tooling needed to carry out difficult formalizations, both in cohomology theories and beyond, I believe these conversations will help me determine some of my research priorities and targets in the future.

Emily Witt As a tenured faculty member who has some familiarity with proof assistants but little experience with Lean, this workshop was fantastic! The organizational structure was great for productivity–a handful of interesting talks throughout the day, with plenty of time in-between to work collaboratively. As a newcomer, I am grateful for the opportunity to give a talk on my goals in entering the mathematical Lean community, which led to, with the help of experts and colleagues, the formalization of local cohomology, a commutative-algebraic notion introduced by Grothendieck. I cannot overstate how much I learned during the workshop about formalization of mathematics in Lean and the state of the math library, through interactions with the welcoming (and patient) organizers and experienced workshop participants.

Scott Morrison The "Formalization of cohomology theories" workshop achieved three difficult things:

- Connecting mathematicians with little or no previous formalization experience with experts who could get them started formalizing material relevant to their reseach.
- Enabling new collaborations, with real work happening in every moment between the talks. (More so than I've seen at almost any other conference I've attended.)
- Relevance across a surprisingly broad range of mathematical topics (unified by the appearance of cohomology, and interest in leveraging new formalization methods for modern mathematics), and relevance to participants from both academia and industry.

I hope this workshop, or variations of it in the form "Formalization of X" for mathematically interesting "X", will become a regular annual event at Banff or elsewhere.

Kevin Buzzard I would like to thank BIRS for hosting us; this was a conference unlike any other I have ever been to. It was broad mathematically and yet still coherent, to the extent that we had analysts, geometers, topologists, algebraists and number theorists interacting in meaningful ways throughout the week.

5 Outcome of the Meeting

The workshop had 51 participants of whom 26 could attend in-person. Attendees were diverse in career status, affiliation and familiarity with formalization. Among the participants were 12 early career researchers. The participant list was international, including members from Australia, China, Europe, the USA, along with 7 Canadians. While the proof assistant Lean received the most attention during the workshop, the presentation showcased the wider variety of proof assistants and computer algebra systems used in research. Thanks to BIRS' technical support, those attending virtually could watch and give presentations, and all attendees could participate in discussions in the Zulip chatroom. The schedule included time for discussions and group work, allowing for productive collaborations in dynamic groups. The wonderful atmosphere provided by BIRS served as a catalyst in the further design and development of formalized mathematics, as reflected in the following two testimonials:

- Sam van Gool: "The BIRS workshop was very stimulating for me. I learned a lot and made new scientific connections, both on the formalization side and on the theory side of things. Being together in the beautiful surroundings of Banff with a relatively small group of experts was very helpful for this. Working in small groups of varying composition, we produced a number of contributions to Lean's mathematical libraries, and I hope that some of the scientific discussions that I had at BIRS will lead to new collaborations in the future."

- Jake Levinson: "Very productive and friendly workshop! The Juniper Hotel was a good location for focused work and interaction. I think the participants all learned a lot and came away with new plans and ideas for expanding the reach of Lean's mathematics library and mathematical formalization more generally."

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