Positive mass theorem and the CR Yamabe equation on 5-dimensional contact spin manifolds

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Short introduction to terminology

On a closed contact (2n + 1)-manifold M, we endow with an almost complex structure J (called CR structure).

- Define the p-mass for (M, J) or its asymp. flat "blowup" $(M\backslash\{p_\infty\}, J, \theta)$ at p_∞ by

$$m(J,\theta) := \lim_{\Lambda\to\infty} \sqrt{-1}n \oint_{S_\Lambda} \sum_{j=1}^{n} \omega_j^j \wedge \theta \wedge (d\theta)^{n-1}$$

  – (a CR analogue of the ADM-mass in GR) where ω_i^j are connection forms wrt (J,θ) and the contact form $\theta = G_{p_\infty}^{2/n}\theta_M$, G_{p_∞} being the Green function at p_∞ for the CR Laplacian L_b.

- Define the CR Yamabe constant $\mathcal{Y}(J)$ by

$$\mathcal{Y}(J) := \inf_\theta \frac{\int_M W_{J,\theta}\theta \wedge (d\theta)^n}{\left(\int_M \theta \wedge (d\theta)^n\right)^{\frac{2n}{2n+2}}} \tag{1}$$

  – where $W_{J,\theta}$ denotes the Tanaka-Webster scalar curvature wrt (J,θ).

The CR Yamabe equation with critical Sobolev exponent

- Let $\theta = u^{2/n}\theta_M$, $u > 0$ in (1). If $0 < \mathcal{Y}(J)$ is attained by u, then u (up to a constant) satisfies

$$L_b u := [(2 + \frac{2}{n})\Delta_b + W]u = u^{1+\frac{2}{n}} \text{ on } M$$

– (called the *CR* Yamabe equation) with minimum energy.

- (M, J) is called embeddable if there is a *CR* embedding

$$\varphi : (M, J) \to (\mathbb{C}^N, J_{\mathbb{C}^N})$$

– i.e. $J_{\mathbb{C}^N} \circ \varphi_* = \varphi_* \circ J$ on the contact bundle.

- **[CMY, 2017]** (PMT for dim 3; based on the Hsiao-Yung solution to $\Box_b$ equation on weighted Sobolev spaces)
  Suppose $\mathcal{Y}(J) > 0$ and J is embeddable ($\cong$ original condition by Yuya Takeuchi). Then
  – (1) $m(J,\theta) \geq 0$;
  – (2) $m(J,\theta) = 0 \implies (M,J) \overset{CR}{\simeq} (S^3, J_{S^3})$.
- **Cor**.: The *CR* Yamabe equation has a solution with minimum energy for (M,J) embeddable.
- **Cor.** (a version of generalized Riemann mapping theorem):
  – Let $\Omega \subset \mathbb{C}^2$ be a $s\psi c$ domain close enough to the unit ball $B^2 \subset \mathbb{C}^2$. Suppose $m(J,\theta) = 0 \implies \Omega$ is biholomorphic to B^2.

Rossi spheres: exotic CR 3-spheres

- **[CMY, 2019(a)]** For $0 \neq |s|$ small, the p-mass of the Rossi spheres $S_s^3 := (S^3, J_{(s)})$ is negative. More precisely,

$$m_s = -18\pi s^2 + o(s^2)$$

– for $s \simeq 0$. The mass is never negative for the Riemannian case.

- **[CMY, 2019(b)]** For $0 \neq |s|$ small,
– (1) the infimum of the *CR* Sobolev quotient of S_s^3 coincides with $\mathcal{Y}(J_{S^3})$, i.e.

$$\mathcal{Y}(J_{(s)}) = \mathcal{Y}(J_{S^3});$$

– (2) $\mathcal{Y}(J_{(s)})$ is not attained ($\Longrightarrow$ the *CR* Yamabe equation for S_s^3 has no solution with minimum energy).

- **[CC, 2021]** (PMT for dim 5 (1)) Let (M, ξ) be a closed, contact spin manifold of dim 5. Suppose J is a spherical CR structure on (M, ξ) with $\mathcal{Y}(J) > 0$. Then
  – (1) $m(J, \theta) \geq 0$;
  – (2) $m(J, \theta) = 0 \implies (M, J) \overset{CR}{\simeq} (S^5, J_{S^5})$.
- **[CC, 2019]** The connected sum is closed within a certain class of spin, spherical 5-manifolds with $\mathcal{Y} > 0$, including $S^5/\mathbb{Z}_p$ (p :odd), $S^4 \times S^1_{(a)}$ $(a > 1)$ and $\mathbb{RP}^5 \# \mathbb{RP}^5$), e.g.

$$m_1(S^5/\mathbb{Z}_{p_1}) \# l_1(S^4 \times S^1_{(a)}) \# m_2(S^5/\mathbb{Z}_{p_2}) \# l_2(\mathbb{RP}^5 \# \mathbb{RP}^5)$$
$$(m_j,\ l_j,\ p_j \in \mathbb{N},\ j = 1, 2,\ p_j : odd,\ j = 1, 2).$$

- **Cor.** Over the above 5-manifolds, the CR Yamabe equation has a solution with minimal energy.

PMT in CR geometry of 5D (2)

- **[CC, 2021]** (PMT for dim 5 (2)) Let (N, J, θ) be an asymp. flat, pseudohermitian and spin manifold of dim 5. Suppose J is spherical and $W_{J,\theta} \geq 0$. Then
  – (1) $m(J, \theta) \geq 0$;
  – (2) $m(J, \theta) = 0 \implies (N, J, \theta)$ is isomorphic to the Heisenberg group $(H_2, \mathring{J}, \mathring{\theta})$.
- PMT (2) $\implies$ PMT (1) by blowing up at p_∞ through $\theta = G_{p_\infty}\theta_M$:

$$(N, J, \theta) = (M \backslash \{p_\infty\}, J, G_{p_\infty}\theta_M)$$

  – $W_{J,\theta} \equiv 0$ in this case.

Weizenbock-type formula

Let $e_1, \cdots, e_{2n}$ be an orthonormal (wrt the Levi metric $d\theta(\cdot, J\cdot)$) frame field of ξ and $e_{n+\beta} = Je_\beta$, $1 \leq \beta \leq n$. Let $\mathbb{S}^\pm$ denote the space of positive/negative spinors on the asymp. flat N (e.g. "blowup" $N = M \setminus \{p_\infty\}$).

- Define the contact Dirac operator $D_\xi : \mathbb{S}^\pm \to \mathbb{S}^\mp$ by

$$D_\xi \psi := \sum_{a=1}^{2n} e_a \nabla_{e_a} \psi$$

- Weitzenbock-type formula:

$$D_\xi^2 = \nabla^* \nabla + W - 2 \sum_{\beta=1}^{n} e_\beta e_{n+\beta} \nabla_T \tag{2}$$

– where T is the Reeb vector field: $\theta(T) = 1$, $d\theta(T, \cdot) = 0$.

The case D=5 (n=2)

- Key algebraic fact for the case D=5 (n=2):

$$\sum_{\beta=1}^{2} e_\beta e_{2+\beta} = e_1 e_3 + e_2 e_4 = 0 \text{ on } S^+. \tag{3}$$

- It follows from (2) and (3) that

$$D_\xi^2 = \nabla^* \nabla + W \text{ on } \mathbb{S}^+ \tag{4}$$

- **[Chiu, 2021]** Suppose (N, J, θ) is an asymp. flat, spherical, spin 5-manifold with $W_{J,\theta} \geq 0$. Let ψ_0 be a constant spinor near ∞. Then there exists a spinor (field) $\psi \in \mathbb{S}^+$ s.t.

$$\begin{aligned} D_\xi^2 \psi &= 0, \\ \psi - \psi_0 &\in S^2_{2,-4+\varepsilon}(\mathbb{S}^+) \end{aligned} \tag{5}$$

– where $S^2_{2,-\eta}(\mathbb{S}^+)$ is a weighted Folland-Stein space.

Proof of $m(J, \theta) \geq 0$

- Applying (4) to a solution ψ to (5), taking inner product with ψ and integrating give

$$\int_N (|\nabla \psi|^2 + W|\psi|^2) dV_\theta = c \cdot m(J, \theta), \ \ c > 0.$$

– in which we pick up the mass from the boundary term at ∞ and other boundary terms go away due to the fast decay rate of $\psi - \psi_0$.

- Either assume $W \geq 0$ or when $N = M \backslash \{p_\infty\}$ is a blowup at p_∞ by taking $\theta = G_{p_\infty} \theta_M$, G_{p_∞} : Green's function of L_b on M, then $W = 0$ on N. In either case $m(J, \theta) \geq 0$.

Characterizing $m(J, \theta) = 0$ (I)

- $m(J, \theta) = 0 \Longrightarrow W \equiv 0$ (in either case)
- $m(J, \theta) = 0 \Longrightarrow$ torsion $A_{\alpha\beta} \equiv 0$
- – Let $J_s := \varphi_s^* J$, φ_s generated by T. Find $u_s > 0$ s.t. $(N, J_s, u_s^{2/n}\theta)$ is asymp. flat with $W_{J_s, u_s^{2/n}\theta} = 0$ and

$$0 \leq m(J_s, u_s^{2/n}\theta) = -C_n \int_N W_{J_s,\theta} u_s dV_\theta$$

– Taking $\frac{d}{ds}|_{s=0}$ gives

$$0 = \frac{d}{ds}|_{s=0} m(J_s, u_s^{2/n}\theta) = 2nC_n \int_N \sum_{\alpha,\beta} |A_{\alpha\beta}|^2 dV_\theta.$$

Characterizing $m(J,\theta) = 0$ (II)

- $m(J,\theta) = 0 \Longrightarrow$ pseudohermitian curvature $R_{\alpha\bar{\beta}\rho\bar{\sigma}} \equiv 0$
- Proof: $A_{\alpha\beta} \equiv 0 \Longrightarrow R_{\alpha\bar{\beta}\rho\bar{\sigma},\gamma} - R_{\alpha\bar{\beta}\gamma\bar{\sigma},\rho} = 0$ (Bianchi id) and $R_{\gamma\bar{\sigma},\sigma} = 0$.; (N,J) spherical $\Longrightarrow 0 =$

$$S_{\alpha\bar{\beta}\rho\bar{\sigma}} = R_{\alpha\bar{\beta}\rho\bar{\sigma}} - \frac{1}{n+2}(R_{\alpha\bar{\beta}}h_{\rho\bar{\sigma}} + R_{\rho\bar{\beta}}h_{\alpha\bar{\sigma}} + \delta_{\alpha}^{\beta}R_{\rho\bar{\sigma}} + \delta_{\rho}^{\beta}R_{\alpha\bar{\sigma}}) \quad (6)$$
$$+\frac{W}{(n+1)(n+2)}(\delta_{\alpha}^{\beta}h_{\rho\bar{\sigma}} + \delta_{\rho}^{\beta}h_{\alpha\bar{\sigma}}).$$

– from which we compute $R_{\alpha\bar{\beta}\rho\bar{\sigma},\gamma} - R_{\alpha\bar{\beta}\gamma\bar{\sigma},\rho}$ (=0) and use $W = 0$. We finally obtain

$$0 = \frac{1}{n+2}(-nR_{\alpha\bar{\sigma},\rho}).$$

– i.e. $R_{\alpha\bar{\sigma}}$ is parallel and hence vanishes since N is asymp. flat. By (6) again we get $R_{\alpha\bar{\beta}\rho\bar{\sigma}} = 0$.

$(N, J, \theta) \simeq (H_2, \mathring{J}, \mathring{\theta})$

- Take $q_0 \in N_\infty := N \backslash N_0$, a simply connected nbhd. By using the developing map, we find a pseudohermitian isomorphism $\Psi (= \text{dev}^{-1}) : \text{dev}(N_\infty) =: V \subset H_2 \rightarrow N_\infty$. Observe that V must be a nbhd of ∞.
- Extend Ψ to a covering map $\tilde{\Psi} : H_2 \rightarrow N$ via the pseudohermitian development.
- Note that V is contained in a fundamental domain. If $\tilde{\Psi}$ is not $1 - 1$, then there are at least two fundamental domains. But the one containing V has ∞ volume while any other one has finite volume. So

$$(H_2, \mathring{J}, \mathring{\theta}) \overset{\tilde{\Psi}}{\simeq} (N, J, \theta).$$

PMT$\Longrightarrow \mathcal{Y}(M, J)$ is attained

- **[CC, 2021]** Let (M, ξ) be a closed, contact spin manifold of dim 5. Suppose J is a spherical CR structure on (M, ξ) with $\mathcal{Y}(M, J) > 0$. Then the CR Yamabe equation has a solution with minimum energy.
- Proof: Test function estimate for J spherical and dim $= 2n + 1 \geq 3$ (Zhongyuan Li, dim $= 2n + 1 \geq 7$ unpublished):

$$E(\phi_\beta) \leq \mathcal{Y}(S^{2n+1}, \hat{J})||\phi_\beta||^2_{2+\frac{2}{n}} - C_n m(J, \theta)\beta^{-2n} + O(\beta^{-2n-1})$$

- $m(J, \theta) > 0 \Longrightarrow \mathcal{Y}(M, J) < \mathcal{Y}(S^{2n+1}, \hat{J}) \overset{Jerison-Lee}{\Longrightarrow} \mathcal{Y}(M, J)$ is attained.

Thanks for your attention