Extremal Eigenvalues for the conformal Laplacian

M. Gursky, joint with Samuel Perez-Ayala (Notre Dame)

Geometric PDE and applications to problems in conformal and CR geometry
IAS(Hangzhou) Workshop

May 18, 2021

Spectrum of the laplacian

Spectrum of the laplacian

(Σ^2, g) closed, two-dimensional Riemannian manifold, $\Delta_g =$ Laplace-Beltrami operator ($\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}$),

Spectrum of the laplacian

(Σ^2, g) closed, two-dimensional Riemannian manifold, $\Delta_g =$ Laplace-Beltrami operator ($\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}$),

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots$$

the eigenvalues of $(-\Delta)$ (counted with multiplicity).

Spectrum of the laplacian

(Σ^2, g) closed, two-dimensional Riemannian manifold, $\Delta_g =$ Laplace-Beltrami operator ($\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}$),

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots$$

the eigenvalues of $(-\Delta)$ (counted with multiplicity). Let

$$[g] = \text{the conformal class of } g = \{e^{2w} g \ : \ w \in C^\infty(\Sigma)\}.$$

Spectrum of the laplacian

(Σ^2, g) closed, two-dimensional Riemannian manifold, $\Delta_g =$ Laplace-Beltrami operator ($\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}$),

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots$$

the eigenvalues of $(-\Delta)$ (counted with multiplicity). Let

$$[g] = \text{the conformal class of } g = \{e^{2w}g \,:\, w \in C^\infty(\Sigma)\}.$$

Definition

The first *conformal eigenvalue* of $(\Sigma, [g])$ is

$$\Lambda_1(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(\tilde{g}) \cdot \text{Area}(\Sigma, \tilde{g}).$$

(We will see that it is always finite; the inf is zero.)

Maximal metrics and bounds for Λ_1

Definition

We say that g is maximal if $\lambda_1(g)\text{Area}(\Sigma, g) = \Lambda_1(\Sigma, [g])$.

Examples

Maximal metrics and bounds for Λ_1

Definition

We say that g is <u>maximal</u> if $\lambda_1(g)\text{Area}(\Sigma, g) = \Lambda_1(\Sigma, [g])$.

<u>Examples</u>

1. Hersch ('70): $\Lambda_1(S^2, [g_0]) = 8\pi$, g maximal iff g has constant curvature.

Maximal metrics and bounds for Λ_1

Definition

We say that g is maximal if $\lambda_1(g)\text{Area}(\Sigma, g) = \Lambda_1(\Sigma, [g])$.

Examples

1. Hersch ('70): $\Lambda_1(S^2, [g_0]) = 8\pi$, g maximal iff g has constant curvature.
2. Yang-Yau ('80): If Σ has genus γ, then $\Lambda_1(\Sigma, [g]) \leq 8\pi \left[\frac{\gamma+3}{2}\right]$.

Maximal metrics and bounds for Λ_1

Definition

We say that g is maximal if $\lambda_1(g)\text{Area}(\Sigma, g) = \Lambda_1(\Sigma, [g])$.

Examples

1. Hersch ('70): $\Lambda_1(S^2, [g_0]) = 8\pi$, g maximal iff g has constant curvature.
2. Yang-Yau ('80): If Σ has genus γ, then $\Lambda_1(\Sigma, [g]) \leq 8\pi \left[\frac{\gamma+3}{2}\right]$.
3. Li-Yau ('82) $\Lambda_1(\mathbb{RP}^2, [g_0]) = 12\pi$, maximal iff constant curvature.

Maximal metrics and bounds for Λ_1

Definition

We say that g is <u>maximal</u> if $\lambda_1(g)\text{Area}(\Sigma, g) = \Lambda_1(\Sigma, [g])$.

Examples

1. Hersch ('70): $\Lambda_1(S^2, [g_0]) = 8\pi$, g maximal iff g has constant curvature.
2. Yang-Yau ('80): If Σ has genus γ, then $\Lambda_1(\Sigma, [g]) \leq 8\pi\left[\frac{\gamma+3}{2}\right]$.
3. Li-Yau ('82) $\Lambda_1(\mathbb{RP}^2, [g_0]) = 12\pi$, maximal iff constant curvature.
4. El Soufi-Ilias-Ros ('96): On flat torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$, gave upper bound for $\Lambda_1(\mathbb{T}^2, [g_\Gamma])$ and characterized maximal metrics (see also Nadirashvili '96).

Extremal metrics

- In general, the functional

$$g \in [\bar{g}] \mapsto \lambda_1(g)\text{Area}(\Sigma, g)$$

is continuous but not differentiable.

Extremal metrics

- In general, the functional

$$g \in [\bar{g}] \mapsto \lambda_1(g)\text{Area}(\Sigma, g)$$

is continuous but not differentiable. However, if $\{g(t)\}_{t\in(-\epsilon,\epsilon)}$ is an analytic (in t) family of metrics with

$$g(t) \in [\bar{g}], \quad g(0) = g_0 \in [\bar{g}],$$

- In general, the functional

$$g \in [\bar{g}] \mapsto \lambda_1(g)\mathrm{Area}(\Sigma, g)$$

is continuous but not differentiable. However, if $\{g(t)\}_{t\in(-\epsilon,\epsilon)}$ is an analytic (in t) family of metrics with

$$g(t) \in [\bar{g}], \quad g(0) = g_0 \in [\bar{g}],$$

then the one-sided derivatives

$$\frac{d}{dt}\lambda_1(g(t))\mathrm{Area}(\Sigma, g(t))\Big|_{t=0^+}, \quad \frac{d}{dt}\lambda_1(g(t))\mathrm{Area}(\Sigma, g(t))\Big|_{t=0^-}$$

exist (Berger '73).

Extremal vs. Maximal

Definition

We say that g_0 is <u>extremal</u> (or <u>C-extremal</u>) if for any analytic deformation $\{g(t)\}_{t\in(-\epsilon,\epsilon)}$ with

$$g(t) \in [\bar{g}], \quad g(0) = g_0 \in [\bar{g}],$$

we have

$$\frac{d}{dt}\lambda_1(g(t))\text{Area}(\Sigma, g(t))\Big|_{t=0^+} \leq 0 \leq \frac{d}{dt}\lambda_1(g(t))\text{Area}(\Sigma, g(t))\Big|_{t=0^-}$$

.

- Note that maximal $\Rightarrow$ extremal, but the converse may not hold.

Extremal metrics and harmonic maps

There is a remarkable connection between extremal metrics and harmonic maps:

Extremal metrics and harmonic maps

There is a remarkable connection between extremal metrics and harmonic maps:

Theorem (Nadirashvili)

If g is extremal, then there is a collection of first eigenfunctions $\{\phi_i\}_{1 \le i \le k}$ such that

$$\sum_{i=1}^{k} \phi_i^2 = 1.$$

Extremal metrics and harmonic maps

There is a remarkable connection between extremal metrics and harmonic maps:

Theorem (Nadirashvili)

If g is extremal, then there is a collection of first eigenfunctions $\{\phi_i\}_{1\le i\le k}$ such that

$$\sum_{i=1}^{k} \phi_i^2 = 1.$$

In particular, $\Phi = (\phi_1, \dots, \phi_k) : \Sigma \to S^{k-1}$ is a harmonic map with constant energy density $|d\Phi|^2 = \lambda_1(g)$.

Extremal metrics and harmonic maps

There is a remarkable connection between extremal metrics and harmonic maps:

Theorem (Nadirashvili)

If g is extremal, then there is a collection of first eigenfunctions $\{\phi_i\}_{1\leq i\leq k}$ such that

$$\sum_{i=1}^{k}\phi_i^2 = 1.$$

In particular, $\Phi = (\phi_1, \ldots, \phi_k) : \Sigma \to S^{k-1}$ is a harmonic map with constant energy density $|d\Phi|^2 = \lambda_1(g)$.

• Note that for maximal metrics, λ_1 is not simple (which is generically the case).

Existence of maximal metrics

Petrides and Nadirashvili-Sire proved the existence of maximal metrics:

Theorem (Petrides '13, Nadirashvili-Sire '15)

Given $(\Sigma, [g])$, then there is a maximal metric $\tilde{g} \in [g]$. Moreover, $\tilde{g}$ is smooth except possibly at finitely many conical singular points.

Existence of maximal metrics

Petrides and Nadirashvili-Sire proved the existence of maximal metrics:

Theorem (Petrides '13, Nadirashvili-Sire '15)

Given $(\Sigma, [g])$, then there is a maximal metric $\tilde{g} \in [g]$. Moreover, $\tilde{g}$ is smooth except possibly at finitely many conical singular points.

Remarks.

1. Parts of Petrides' proof rely on earlier work of Kokarev.

Existence of maximal metrics

Petrides and Nadirashvili-Sire proved the existence of maximal metrics:

Theorem (Petrides '13, Nadirashvili-Sire '15)

Given $(\Sigma, [g])$, then there is a maximal metric $\tilde{g} \in [g]$. Moreover, $\tilde{g}$ is smooth except possibly at finitely many conical singular points.

Remarks.

1. Parts of Petrides' proof rely on earlier work of Kokarev.
2. The regularity statement follows from the regularity theory of harmonic maps.

Existence of maximal metrics

Petrides and Nadirashvili-Sire proved the existence of maximal metrics:

Theorem (Petrides '13, Nadirashvili-Sire '15)

Given $(\Sigma, [g])$, then there is a maximal metric $\tilde{g} \in [g]$. Moreover, $\tilde{g}$ is smooth except possibly at finitely many conical singular points.

Remarks.

1. Parts of Petrides' proof rely on earlier work of Kokarev.
2. The regularity statement follows from the regularity theory of harmonic maps.
3. A key property used throughout the proofs (and proofs of related results) is conformal invariance of the Laplacian in two dimensions (and the conformal invariance of the harmonic map equation).

Higher dimensions

• In higher dimensions, the Laplace-Beltrami operator is not conformally invariant, but there are conformally invariant operators (of the form $(-\Delta)^p + (\text{l.o.t.})$.

Higher dimensions

• In higher dimensions, the Laplace-Beltrami operator is not conformally invariant, but there are conformally invariant operators (of the form $(-\Delta)^p + (\text{l.o.t.})$.

Example. If (M, g) is n-dimensional with $n \geq 3$, then the *conformal Laplacian* is

$$L_g = -\Delta_g + c_n R_g,$$

where

$$c_n = \frac{4(n-1)}{(n-2)}.$$

Higher dimensions

• In higher dimensions, the Laplace-Beltrami operator is not conformally invariant, but there are conformally invariant operators (of the form $(-\Delta)^p + (\text{l.o.t.})$.

Example. If (M, g) is n-dimensional with $n \geq 3$, then the *conformal Laplacian* is

$$L_g = -\Delta_g + c_n R_g,$$

where

$$c_n = \frac{4(n-1)}{(n-2)}.$$

L is conformally invariant in the sense that if $\tilde{g} = u^{\frac{4}{n-2}} g$, then

$$L_{\tilde{g}} \phi = u^{-\frac{n+2}{n-2}} L_g(u\phi).$$

• In higher dimensions, the Laplace-Beltrami operator is not conformally invariant, but there are conformally invariant operators (of the form $(-\Delta)^p + (\text{l.o.t.})$.

Example. If (M, g) is n-dimensional with $n \geq 3$, then the *conformal Laplacian* is

$$L_g = -\Delta_g + c_n R_g,$$

where

$$c_n = \frac{4(n-1)}{(n-2)}.$$

L is conformally invariant in the sense that if $\tilde{g} = u^{\frac{4}{n-2}} g$, then

$$L_{\tilde{g}}\phi = u^{-\frac{n+2}{n-2}} L_g(u\phi).$$

Let $\lambda_1(L_g) < \lambda_2(L_g) \leq \lambda_3(L_g) \leq \cdots$ denote the eigenvalues of L_g.

The conformal laplacian

There are several conformal invariants associated to the spectrum of L_g:

The conformal laplacian

There are several conformal invariants associated to the spectrum of L_g:

- The sign of $\lambda_1(L_g)$ is a conformal invariant and agrees with the sign of the Yamabe invariant

$$Y(M^n,[g]) := \inf_{u\in W^{1,2}\setminus\{0\}} \frac{\int_M u\, L_g u\, dv_g}{\left(\int_M |u|^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}.$$

The conformal laplacian

There are several conformal invariants associated to the spectrum of L_g:

- The sign of $\lambda_1(L_g)$ is a conformal invariant and agrees with the sign of the Yamabe invariant

$$Y(M^n,[g]) := \inf_{u\in W^{1,2}\setminus\{0\}} \frac{\int_M u\, L_g\, u\, dv_g}{\left(\int_M |u|^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}.$$

- The dimension of $\ker L_g$ is a conformal invariant.

The conformal laplacian

There are several conformal invariants associated to the spectrum of L_g:

- The sign of $\lambda_1(L_g)$ is a conformal invariant and agrees with the sign of the Yamabe invariant

$$Y(M^n,[g]) := \inf_{u\in W^{1,2}\setminus\{0\}} \frac{\int_M u\, L_g\, u\, dv_g}{\left(\int_M |u|^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}.$$

- The dimension of $\ker L_g$ is a conformal invariant.
- The number of negative eigenvalues of L_g, $\nu([g])$, is also a conformal invariant; cf. Canzani-Gover-Jakobson-Ponge '14.

Variational Properties of L

• To see the issues that arise when studying the variational properties of L, first assume that $(M,[g])$ has $\lambda_1(L_g) > 0$.

Variational Properties of L

• To see the issues that arise when studying the variational properties of L, first assume that $(M, [g])$ has $\lambda_1(L_g) > 0$.

• In analogy with the case of surfaces, we might be tempted to study the obvious generalization

$$\Lambda_1(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(L_{\tilde{g}}) \mathrm{Vol}(M, \tilde{g})^{\frac{2}{n}}.$$

Variational Properties of L

• To see the issues that arise when studying the variational properties of L, first assume that $(M, [g])$ has $\lambda_1(L_g) > 0$.

• In analogy with the case of surfaces, we might be tempted to study the obvious generalization

$$\Lambda_1(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(L_{\tilde{g}}) \mathrm{Vol}(M, \tilde{g})^{\frac{2}{n}}.$$

However, by a result of Ammann-Jammes '08, the supremum is always $+\infty$ (Korevaar '93 showed that $\lambda_1(-\Delta) \cdot V^{2/n}$ *is* bounded).

Variational Properties of L

• To see the issues that arise when studying the variational properties of L, first assume that $(M, [g])$ has $\lambda_1(L_g) > 0$.

• In analogy with the case of surfaces, we might be tempted to study the obvious generalization

$$\Lambda_1(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(L_{\tilde{g}}) \mathrm{Vol}(M, \tilde{g})^{\frac{2}{n}}.$$

However, by a result of Ammann-Jammes '08, the supremum is always $+\infty$ (Korevaar '93 showed that $\lambda_1(-\Delta) \cdot V^{2/n}$ *is* bounded).

• Interestingly, they showed that the issue is the *order* of L. Roughly, they showed that if the order of a conformally invariant operator is less than the dimension, then the sup is always infinite.

Variational Properties of L, cont.

- What about $\inf_{[g]} \lambda_1(L)\, V^{\frac{2}{n}}$?

Variational Properties of L, cont.

• What about $\inf_{[g]} \lambda_1(L)\, V^{\frac{2}{n}}$? It turns out that this is equivalent to solving the Yamabe problem.

Variational Properties of L, cont.

• What about $\inf_{[g]} \lambda_1(L)\, V^{\frac{2}{n}}$? It turns out that this is equivalent to solving the Yamabe problem. To see this, recall

$$\lambda_1(L_{\tilde{g}}) = \inf_{\phi \in W^{1,2}} \frac{\int \phi\, L_{\tilde{g}} \phi\, dv_{\tilde{g}}}{\int \phi^2\, dv_{\tilde{g}}}$$

Variational Properties of L, cont.

• What about $\inf_{[g]} \lambda_1(L)\, V^{\frac{2}{n}}$? It turns out that this is equivalent to solving the Yamabe problem. To see this, recall

$$\lambda_1(L_{\tilde{g}}) = \inf_{\phi \in W^{1,2}} \frac{\int \phi\, L_{\tilde{g}} \phi\, dv_{\tilde{g}}}{\int \phi^2\, dv_{\tilde{g}}}$$

If $\tilde{g} = u^{\frac{4}{n-2}} g$, then using the conformal invariance of L this can also be written

Variational Properties of L, cont.

• What about $\inf_{[g]} \lambda_1(L)\, V^{\frac{2}{n}}$? It turns out that this is equivalent to solving the Yamabe problem. To see this, recall

$$\lambda_1(L_{\tilde{g}}) = \inf_{\phi \in W^{1,2}} \frac{\int \phi\, L_{\tilde{g}} \phi \, dv_{\tilde{g}}}{\int \phi^2 \, dv_{\tilde{g}}}$$

If $\tilde{g} = u^{\frac{4}{n-2}} g$, then using the conformal invariance of L this can also be written

$$\lambda_1(L_{\tilde{g}}) = \inf_{\phi \in W^{1,2}} \frac{\int (u\phi)\, L_g (u\phi)\, dv_g}{\int (u\phi)^2\, u^{\frac{4}{n-2}}\, dv_g}$$

Variational Properties of L, cont.

• What about $\inf_{[g]} \lambda_1(L)\, V^{\frac{2}{n}}$? It turns out that this is equivalent to solving the Yamabe problem. To see this, recall

$$\lambda_1(L_{\tilde{g}}) = \inf_{\phi \in W^{1,2}} \frac{\int \phi\, L_{\tilde{g}} \phi \, dv_{\tilde{g}}}{\int \phi^2 \, dv_{\tilde{g}}}$$

If $\tilde{g} = u^{\frac{4}{n-2}} g$, then using the conformal invariance of L this can also be written

$$\lambda_1(L_{\tilde{g}}) = \inf_{\phi \in W^{1,2}} \frac{\int (u\phi)\, L_g (u\phi)\, dv_g}{\int (u\phi)^2\, u^{\frac{4}{n-2}} dv_g}$$

$$= \inf_{\psi \in W^{1,2}} \frac{\int \psi\, L_g \psi \, dv_g}{\int \psi^2\, u^{\frac{4}{n-2}} dv_g}$$

Variational Properties of L, cont.

Since we'll see this again, denote

$$\mathcal{R}_g^u(\psi) = \frac{\int \psi\, L_g \psi\, dv_g}{\int \psi^2\, u^{N-2} dv_g} \quad (N = \frac{2n}{n-2})$$

.

Variational Properties of L, cont.

Since we'll see this again, denote

$$\mathcal{R}_g^u(\psi) = \frac{\int \psi \, L_g \psi \, dv_g}{\int \psi^2 \, u^{N-2} dv_g} \quad (N = \frac{2n}{n-2})$$

. By Hölder's inequality,

$$\mathcal{R}_g^u(\psi) \geq \frac{\int \psi \, L_g \psi \, dv_g}{\left(\int |\psi|^N \, dv_g\right)^{2/N} \left(\int u^N \, dv_g\right)^{2/n}}$$

,

Variational Properties of L, cont.

Since we'll see this again, denote

$$\mathcal{R}_g^u(\psi) = \frac{\int \psi\, L_g \psi \, dv_g}{\int \psi^2 \, u^{N-2} dv_g} \quad (N = \frac{2n}{n-2})$$

. By Hölder's inequality,

$$\mathcal{R}_g^u(\psi) \geq \frac{\int \psi\, L_g \psi \, dv_g}{\left(\int |\psi|^N \, dv_g \right)^{2/N} \left(\int u^N \, dv_g \right)^{2/n}}$$

,

$$\Rightarrow \quad \mathrm{Vol}(\tilde{g})^{2/n} \lambda_1(L_{\tilde{g}}) \geq \; Y(M, [g]),$$

and equality holds iff u is constant and $\tilde{g}$ is a Yamabe metric.

$\lambda_2(L_g)$

• In light of these facts, Amman-Humbert '05 defined the *second Yamabe invariant* by

$$\mu_2(M,[g]) = \inf_{\tilde{g}\in[g]} \lambda_2(L_{\tilde{g}})\,\mathrm{Vol}(\tilde{g})^{2/n}.$$

$\lambda_2(L_g)$

• In light of these facts, Amman-Humbert '05 defined the *second Yamabe invariant* by

$$\mu_2(M,[g]) = \inf_{\tilde{g}\in[g]} \lambda_2(L_{\tilde{g}})\,\mathrm{Vol}(\tilde{g})^{2/n}.$$

They showed that μ_2 is <u>never</u> attained by a smooth metric.

$\lambda_2(L_g)$

• In light of these facts, Amman-Humbert '05 defined the *second Yamabe invariant* by

$$\mu_2(M,[g]) = \inf_{\tilde{g}\in[g]} \lambda_2(L_{\tilde{g}})\,\mathrm{Vol}(\tilde{g})^{2/n}.$$

They showed that μ_2 is never attained by a smooth metric.

• Instead, one has to consider 'generalized conformal metrics' defined by

$$\tilde{g} = u^{N-2}g, \quad u \in L^N_+ = \{u \in L^N \,:\, u \geq 0 \text{ } a.e., u \neq 0\} \quad (N = \frac{2n}{n-2})$$

$\lambda_2(L_g)$

• In light of these facts, Amman-Humbert '05 defined the *second Yamabe invariant* by

$$\mu_2(M,[g]) = \inf_{\tilde{g}\in[g]} \lambda_2(L_{\tilde{g}})\,\mathrm{Vol}(\tilde{g})^{2/n}.$$

They showed that μ_2 is <u>never</u> attained by a smooth metric.

• Instead, one has to consider 'generalized conformal metrics' defined by

$$\tilde{g} = u^{N-2}g, \quad u \in L^N_+ = \{u \in L^N \,:\, u \geq 0 \; a.e., u \neq 0\} \quad (N = \frac{2n}{n-2})$$

• In this context, $\lambda_2(L_g)$ is defined via the minimax characterization:

$$\lambda_2(L_{\tilde{g}}) = \inf_{\Sigma\subset W^{1,2}} \sup_{\psi\in\Sigma} \mathcal{R}^u_g(\psi),$$

where $\Sigma \subset W^{1,2}$ is a two-dimensional subspace*.

$\lambda_2(L_g)$

• A function which attains the minimax is a *generalized (second) eigenfunction*, and satisfies the equation

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

$\lambda_2(L_g)$

• A function which attains the minimax is a *generalized (second) eigenfunction*, and satisfies the equation

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

• If $u > 0$ and smooth, let $\tilde{g} = u^{N-2} g$ and suppose $\widehat{\phi}_2$ a second eigenfunction for $L_{\tilde{g}}$. Then

$$L_{\tilde{g}} \widehat{\phi}_2 = \lambda_2(L_{\tilde{g}})\, \widehat{\phi}_2.$$

$\lambda_2(L_g)$

• A function which attains the minimax is a *generalized (second) eigenfunction*, and satisfies the equation

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

• If $u > 0$ and smooth, let $\tilde{g} = u^{N-2} g$ and suppose $\widehat{\phi}_2$ a second eigenfunction for $L_{\tilde{g}}$. Then

$$L_{\tilde{g}} \widehat{\phi}_2 = \lambda_2(L_{\tilde{g}})\, \widehat{\phi}_2.$$

By conformal invariance, this implies

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2},$$

where $\phi_2 = u\widehat{\phi}_2$. In particular, generalized eigenfunctions are classical eigenfunctions when the metric is smooth.

$\mu_2(M, [g])$

Using concentration-compactness arguments, Ammann-Humbert proved

$\mu_2(M, [g])$

Using concentration-compactness arguments, Ammann-Humbert proved

Theorem (Ammann-Humbert)

Assume $Y(M, [g]) > 0$. (i) There is a dimensional constant κ_n such that if

$$\mu_2(M, [g]) < \kappa_n,$$

then $\mu_2(M, [g])$ is attained by a generalized conformal metric $\tilde{g} = u^{N-2} g$, with $u \in L^N_+$.

$\mu_2(M, [g])$

Using concentration-compactness arguments, Ammann-Humbert proved

Theorem (Ammann-Humbert)

Assume $Y(M, [g]) > 0$. (i) There is a dimensional constant κ_n such that if

$$\mu_2(M, [g]) < \kappa_n,$$

then $\mu_2(M, [g])$ is attained by a generalized conformal metric $\tilde{g} = u^{N-2} g$, with $u \in L^N_+$.
(ii) Moreover, there is a second (generalized) eigenfunction ϕ_2 such that

$$\phi_2^2 = u^2.$$

$\mu_2(M, [g])$

Using concentration-compactness arguments, Ammann-Humbert proved

Theorem (Ammann-Humbert)

Assume $Y(M, [g]) > 0$. (i) There is a dimensional constant κ_n such that if

$$\mu_2(M, [g]) < \kappa_n,$$

then $\mu_2(M, [g])$ is attained by a generalized conformal metric $\tilde{g} = u^{N-2} g$, with $u \in L^N_+$.
(ii) Moreover, there is a second (generalized) eigenfunction ϕ_2 such that

$$\phi_2^2 = u^2.$$

Consequently $\phi_2 \in C^{3,\alpha}$ is a nodal solution of the Yamabe equation

$$L_g \phi_2 = \mu_2(M, [g]) \phi_2 |\phi_2|^{N-2}.$$

Remarks on μ_2

• A somewhat surprising aspect of the Ammann-Humbert result is that the minimal (generalized) eigenvalue is simple. This is a significant contrast with the case of surfaces.

Remarks on μ_2

• A somewhat surprising aspect of the Ammann-Humbert result is that the minimal (generalized) eigenvalue is simple. This is a significant contrast with the case of surfaces.

• In fact, the condition

$$u^2 = \phi_2^2$$

is actually a generalization of the condition we saw for surfaces (sum of the squares of the eigenfunctions is constant). In the higher-dimensional case,

Remarks on μ_2

• A somewhat surprising aspect of the Ammann-Humbert result is that the minimal (generalized) eigenvalue is simple. This is a significant contrast with the case of surfaces.

• In fact, the condition

$$u^2 = \phi_2^2$$

is actually a generalization of the condition we saw for surfaces (sum of the squares of the eigenfunctions is constant). In the higher-dimensional case,

1. there is only one eigenfunction, and

Remarks on μ_2

• A somewhat surprising aspect of the Ammann-Humbert result is that the minimal (generalized) eigenvalue is simple. This is a significant contrast with the case of surfaces.

• In fact, the condition

$$u^2 = \phi_2^2$$

is actually a generalization of the condition we saw for surfaces (sum of the squares of the eigenfunctions is constant). In the higher-dimensional case,

1. there is only one eigenfunction, and
2. the factor u^2 (vs 1) simply reflects the difference in conformal weights between Δ_g in two dimensions and L_g in higher dimensions.

Remarks on μ_2

• A somewhat surprising aspect of the Ammann-Humbert result is that the minimal (generalized) eigenvalue is simple. This is a significant contrast with the case of surfaces.

• In fact, the condition

$$u^2 = \phi_2^2$$

is actually a generalization of the condition we saw for surfaces (sum of the squares of the eigenfunctions is constant). In the higher-dimensional case,

1. there is only one eigenfunction, and
2. the factor u^2 (vs 1) simply reflects the difference in conformal weights between Δ_g in two dimensions and L_g in higher dimensions.

• But...what happened to the connection to harmonic maps?

The case of negative Yamabe invariant

- If $\lambda_1(L_g) < 0$, let $\nu([g]) = \#$ of negative eigenvalues of L_g.

The case of negative Yamabe invariant

- If $\lambda_1(L_g) < 0$, let $\nu([g]) = \#$ of negative eigenvalues of L_g. If $\nu([g]) > 1$, then $\lambda_2(L_g) < 0$.

The case of negative Yamabe invariant

- If $\lambda_1(L_g) < 0$, let $\nu([g]) = \#$ of negative eigenvalues of L_g. If $\nu([g]) > 1$, then $\lambda_2(L_g) < 0$.

- If $\tilde{g} \in [g]$, then

$$\lambda_1(\tilde{g})\,\mathrm{Vol}(\tilde{g})^{\frac{2}{n}} \leq Y(M, [g]),$$

with equality iff $\tilde{g}$ is the unique Yamabe metric (up to scaling).

The case of negative Yamabe invariant

- If $\lambda_1(L_g) < 0$, let $\nu([g]) = \#$ of negative eigenvalues of L_g. If $\nu([g]) > 1$, then $\lambda_2(L_g) < 0$.

- If $\tilde{g} \in [g]$, then

$$\lambda_1(\tilde{g})\,\mathrm{Vol}(\tilde{g})^{\frac{2}{n}} \leq Y(M, [g]),$$

with equality iff $\tilde{g}$ is the unique Yamabe metric (up to scaling).

- It is not difficult to show that

$$\mu_2(M, [g]) = \inf_{\tilde{g} \in [g]} \lambda_2(\tilde{g})\,\mathrm{Vol}(\tilde{g})^{\frac{2}{n}} = -\infty.$$

The case of negative Yamabe invariant

• If $\lambda_1(L_g) < 0$, let $\nu([g]) = \#$ of negative eigenvalues of L_g. If $\nu([g]) > 1$, then $\lambda_2(L_g) < 0$.

• If $\tilde{g} \in [g]$, then

$$\lambda_1(\tilde{g})\,\mathrm{Vol}(\tilde{g})^{\frac{2}{n}} \leq Y(M, [g]),$$

with equality iff $\tilde{g}$ is the unique Yamabe metric (up to scaling).

• It is not difficult to show that

$$\mu_2(M, [g]) = \inf_{\tilde{g} \in [g]} \lambda_2(\tilde{g})\,\mathrm{Vol}(\tilde{g})^{\frac{2}{n}} = -\infty.$$

• Consequently, in the case of negative Yamabe invariant it is natural to consider

$$\overline{\mu}_2(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_2(L_{\tilde{g}})\,\mathrm{Vol}(\tilde{g})^{2/n}.$$

Statement of the result

Theorem (G-Perez-Ayala, '20)

Assume $\nu([g]) \geq 2$ and $0 \notin \mathrm{Spec}(L_g)$.

Statement of the result

Theorem (G-Perez-Ayala, '20)

Assume $\nu([g]) \geq 2$ and $0 \notin \mathrm{Spec}(L_g)$. Then there is a (possibly generalized) maximal conformal metric $\tilde{g} = \bar{u}^{N-2} g$ with $\bar{u} \in \mathrm{Lip} \cap C^\infty(M^n \setminus \{\bar{u} = 0\})$.

Statement of the result

Theorem (G-Perez-Ayala, '20)

Assume $\nu([g]) \geq 2$ and $0 \notin \mathrm{Spec}(L_g)$. Then there is a (possibly generalized) maximal conformal metric $\tilde{g} = \bar{u}^{N-2} g$ with $\bar{u} \in \mathrm{Lip} \cap C^\infty(M^n \setminus \{\bar{u} = 0\})$.

Moreover, for any maximizer $\bar{u} \in L^N_+(M^n, g)$,

Statement of the result

Theorem (G-Perez-Ayala, '20)

Assume $\nu([g]) \geq 2$ and $0 \notin \mathrm{Spec}(L_g)$. Then there is a (possibly generalized) maximal conformal metric $\tilde{g} = \bar{u}^{N-2} g$ with $\bar{u} \in \mathrm{Lip} \cap C^\infty(M^n \setminus \{\bar{u} = 0\})$.

Moreover, for any maximizer $\bar{u} \in L^N_+(M^n, g)$, there exists a collection $\{\bar{\phi}_i\}_{i=1}^k \in C^{2,\alpha}(M^n)$ of second generalized eigenfunctions satisfying

$$\bar{u}^2 = \sum_{i=1}^k \bar{\phi}_i^2.$$

Here $1 \leq k \leq \dim E_2(\bar{u})$, where $E_2(\bar{u})$ is the space of generalized eigenfunctions corresponding to $\lambda_2(\bar{u})$.

A Dichotomy

Corollary

Let $\tilde{g} = \bar{u}^{N-2} g$ be a maximal metric as in the preceding. We have the following two cases:

A Dichotomy

Corollary

Let $\tilde{g} = \bar{u}^{N-2} g$ be a maximal metric as in the preceding. We have the following two cases:

1. If $k = 1$, then $\bar{u} = |\bar{\phi}|$ on M^n, and $\bar{\phi}$ is a nodal solution of

$$L_g \bar{\phi} = \lambda_2(\bar{u}) |\bar{\phi}|^{\frac{4}{n-2}} \bar{\phi}.$$

A Dichotomy

Corollary

Let $\tilde{g} = \bar{u}^{N-2} g$ be a maximal metric as in the preceding. We have the following two cases:

1. If $k = 1$, then $\bar{u} = |\bar{\phi}|$ on M^n, and $\bar{\phi}$ is a nodal solution of

$$L_g \bar{\phi} = \lambda_2(\bar{u}) |\bar{\phi}|^{\frac{4}{n-2}} \bar{\phi}.$$

2. If $k > 1$, then the map

$$\bar{U} := (\bar{\phi}_1/\bar{u}, \cdots, \bar{\phi}_k/\bar{u}) : (M^n \setminus \{\bar{u} = 0\}, \bar{u}^{N-2} g) \longrightarrow (\mathbb{S}^{k-1}, g_{\text{round}})$$

   defines a harmonic map.

Examples

- Crucially, we can show that both possibilities actually occur:

Examples

• Crucially, we can show that both possibilities actually occur: there are conformal classes for which the maximal metric defines a nodal solution of the Yamabe problem ($k = 1$), and there conformal classes for which the maximal metric defines a harmonic map ($k \geq 2$).

Examples

• Crucially, we can show that both possibilities actually occur: there are conformal classes for which the maximal metric defines a nodal solution of the Yamabe problem ($k = 1$), and there conformal classes for which the maximal metric defines a harmonic map ($k \geq 2$).

• For the former case, suppose $\nu([g]) = 2$. Since $\lambda_1(L)$ is simple, the multiplicity of $\lambda_2(L)$ must be one. Therefore, $k = 1$ in the Theorem above and the maximal metric must define a nodal solution of the Yamabe equation.

Example of a Harmonic Map

Theorem

Let (H, g) be a closed Riemannian manifold with constant negative scalar curvature, suitably normalized. Then the product metric $(M, g) = (H \times S^m, h + g_0)$, where g_0 is the round metric, is maximal in its conformal class. In particular, eigenfunctions $\{\psi_1, \ldots, \psi_{m+1}\}$ for the laplacian on the S^m-factor are eigenfunctions for $\lambda_2(L_g)$, and define a harmonic map

$$\Psi = (\psi_1, \ldots, \psi_{m+1}) : M \to S^m,$$

given by projection onto the S^m-factor.

- In particular, this gives an example for which $k = m + 1$.

Remarks

• Recall in the work of Ammann-Humbert, a minimizer for $\lambda_2(L)$ is always simple, while in the preceding example the multiplicity of λ_2 is $m+1$.

Remarks

• Recall in the work of Ammann-Humbert, a minimizer for $\lambda_2(L)$ is always simple, while in the preceding example the multiplicity of λ_2 is $m+1$.

• Again in contrast to the work of Amman-Humbert in the positive case, in the preceding Theorem the maximal metric is *smooth*.

Remarks

• Recall in the work of Ammann-Humbert, a minimizer for $\lambda_2(L)$ is always simple, while in the preceding example the multiplicity of λ_2 is $m+1$.

• Again in contrast to the work of Amman-Humbert in the positive case, in the preceding Theorem the maximal metric is *smooth*.

• Another surprising aspect of this example is that the product metric is a Yamabe metric, hence is simultaneously maximal for $\lambda_1(L)$. This is remarkably different from the case of the Laplace operator on surfaces, where it is known that metrics cannot maximize consecutive eigenvalues (cf. El Soufi)

Sketch of the proof

• One of the main technical issues that arises when trying to maximize λ_2 is the lack of control of λ_1 (this is absent in the case of positive Yamabe invariant).

Sketch of the proof

• One of the main technical issues that arises when trying to maximize λ_2 is the lack of control of λ_1 (this is absent in the case of positive Yamabe invariant). Therefore, we regularize the problem in the following way:

Sketch of the proof

• One of the main technical issues that arises when trying to maximize λ_2 is the lack of control of λ_1 (this is absent in the case of positive Yamabe invariant). Therefore, we regularize the problem in the following way: for $\epsilon > 0$, let

$$\mathcal{D}_\epsilon = \{u \in L^N_+ \, : \, \int u^{-\epsilon} \, dv_g < \infty\}, \quad N = \frac{2n}{n-2}$$

Sketch of the proof

• One of the main technical issues that arises when trying to maximize λ_2 is the lack of control of λ_1 (this is absent in the case of positive Yamabe invariant). Therefore, we regularize the problem in the following way: for $\epsilon > 0$, let

$$\mathcal{D}_\epsilon = \{u \in L^N_+ : \int u^{-\epsilon}\, dv_g < \infty\}, \quad N = \frac{2n}{n-2}$$

• If $u \in \mathcal{D}_\epsilon$, then

$$\lambda_1(u) > -\infty,$$

where

$$\lambda_1(u) = \lambda_1(L_{u^{N-2}g}).$$

Sketch of the proof

• One of the main technical issues that arises when trying to maximize λ_2 is the lack of control of λ_1 (this is absent in the case of positive Yamabe invariant). Therefore, we regularize the problem in the following way: for $\epsilon > 0$, let

$$\mathcal{D}_\epsilon = \{u \in L^N_+ : \int u^{-\epsilon}\, dv_g < \infty\}, \quad N = \frac{2n}{n-2}$$

• If $u \in \mathcal{D}_\epsilon$, then

$$\lambda_1(u) > -\infty,$$

where

$$\lambda_1(u) = \lambda_1(L_{u^{N-2}g}).$$

• We define our regularized functional $F_\epsilon : \mathcal{D}_\epsilon \to \mathbb{R}$ in the following way:

Sketch of the proof, cont.

$$F_\epsilon(u) = \lambda_2(u) \Big(\int u^N \, dv_g \Big)^{\frac{2}{n}} - \Big(\int u^{-\epsilon} \, dv_g \Big) \Big(\int u^N \, dv_g \Big)^{\frac{\epsilon}{N}},$$

This functional is scale-invariant.

Sketch of the proof, cont.

$$F_\epsilon(u) = \lambda_2(u)\Big(\int u^N \, dv_g\Big)^{\frac{2}{n}} - \Big(\int u^{-\epsilon}\, dv_g\Big)\Big(\int u^N \, dv_g\Big)^{\frac{\epsilon}{N}},$$

This functional is scale-invariant.

Theorem

For each $\epsilon > 0$, there is a $u_\epsilon \in \mathcal{D}_\epsilon$, normalized so that $\|u_\epsilon\|_{L^N} = 1$, which maximizes F_ϵ.

Sketch of the proof, cont.

$$F_\epsilon(u) = \lambda_2(u)\Big(\int u^N\,dv_g\Big)^{\frac{2}{n}} - \Big(\int u^{-\epsilon}\,dv_g\Big)\Big(\int u^N\,dv_g\Big)^{\frac{\epsilon}{N}},$$

This functional is scale-invariant.

Theorem

For each $\epsilon > 0$, there is a $u_\epsilon \in \mathcal{D}_\epsilon$, normalized so that $\|u_\epsilon\|_{L^N} = 1$, which maximizes F_ϵ. Moreover, there is a constant $\gamma > 0$ and a set of (generalized) eigenfunctions associated to $\lambda_2(u_\epsilon)$ such that

$$u_\epsilon^2 = \sum_{i=1}^{k} (\phi_i^\epsilon)^2 + \epsilon\gamma u_\epsilon^{2-N-\epsilon}.$$

Sketch of the proof, cont.

- Recall the generalized second eigenfunction equation:

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

Sketch of the proof, cont.

• Recall the generalized second eigenfunction equation:

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

• Using the regularized equation, one can obtain estimates for the generalized eigenfunctions

$$\|\phi_i^\epsilon\|_{C^{1,\alpha}} \leq C.$$

Sketch of the proof, cont.

• Recall the generalized second eigenfunction equation:

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

• Using the regularized equation, one can obtain estimates for the generalized eigenfunctions

$$\|\phi_i^{\epsilon}\|_{C^{1,\alpha}} \leq C.$$

Also, we have a key *a priori* bound

$$\int u_{\epsilon}^{-N-\epsilon}\, dv_g \leq C.$$

Sketch of the proof, cont.

• Recall the generalized second eigenfunction equation:

$$L_g \phi_2 = \lambda_2(L_{\tilde{g}})\, \phi_2\, u^{N-2}.$$

• Using the regularized equation, one can obtain estimates for the generalized eigenfunctions

$$\|\phi_i^\epsilon\|_{C^{1,\alpha}} \leq C.$$

Also, we have a key *a priori* bound

$$\int u_\epsilon^{-N-\epsilon}\, dv_g \leq C.$$

• Using these estimates (and others), we can take the limit $\epsilon \to 0$ and obtain a (generalized) metric that maximizes λ_2.

Further questions, ongoing work

• Question (M. Karpukhin): Do harmonic maps give rise to extremal eigenvalues?

Further questions, ongoing work

• Question (M. Karpukhin): Do harmonic maps give rise to extremal eigenvalues?

• In continuing work with Perez-Ayala, we are studying a conformally invariant Steklov problem.

Further questions, ongoing work

• Question (M. Karpukhin): Do harmonic maps give rise to extremal eigenvalues?

• In continuing work with Perez-Ayala, we are studying a conformally invariant Steklov problem.

Thank you.