Thin-shell concentration on Orlicz balls

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Joint work with J. Prochno

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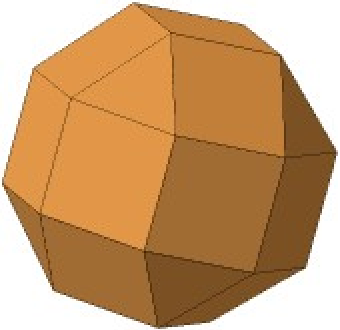
Convex bodies

- $K \subset \mathbb{R}^n$ is called a convex body if it is convex, compact and has non-empty interior.



Isotropic bodies

A convex body $K \subseteq \mathbb{R}^n$ is isotropic if it has volume 1 and

- $\displaystyle\int_K x dx = 0$ (centered at 0)
- $\displaystyle\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \forall \theta \in S^{n-1}$.

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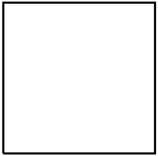

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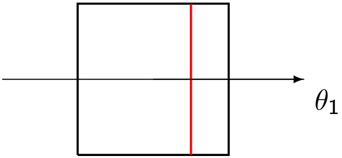

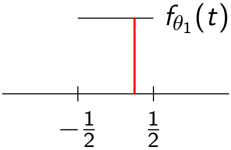

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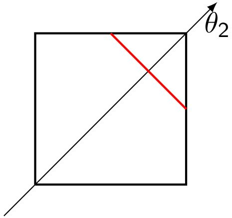

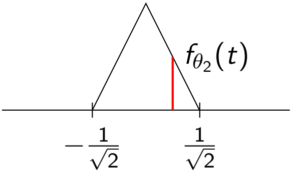

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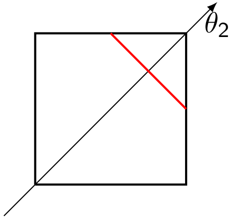

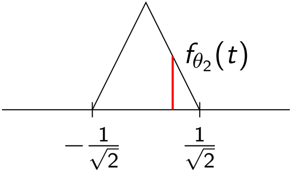

K is isotropic if all the $\langle X, \theta \rangle$ are centered and have the same variance.

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- $\displaystyle nL_K^2 = \min\left\{\frac{1}{|TK|^{1+\frac{2}{n}}}\int_{TK} |x|^2 dx \;:\; T \in GL(n)\right\}.$

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There exists an **absolute** constant C such that for every $K \subseteq \mathbb{R}^n$

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- True for 1-unconditional bodies, zonoids, unit balls of finite dimensional Schätten classes...

Orlicz balls

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- $\frac{B_M^n(nR)}{|B_M^n(nR)|^{\frac{1}{n}}}$ is isotropic and $L_{B_M^n(nR)} \leq C$.

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Given an Orlicz function $M : \mathbb{R} \to \mathbb{R}$ and $R > 0$,

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- Z denotes a centered random variable with density p_M with respect to the Lebesgue measure.

$$\mathbb{E}[M(Z)] = \varphi_M'(\alpha_*) = R \quad \mathrm{Var}[M(Z)] = \varphi_M''(\alpha_*)$$

Orlicz balls

Theorem (Kabluchko, Prochno, 2021)

Let M be an Orlicz function and $R > 0$. Then

$$\lim_{n\to\infty} |B_M^n(nR)|^{\frac{1}{n}} = e^{\varphi_M(\alpha_*)-\alpha_* R}.$$

Furthermore,

$$|B_M^n(nR)| = \frac{e^{n(\varphi_M(\alpha_*)-\alpha^* R)}}{|\alpha_*|\sqrt{2\pi n\varphi_M''(\alpha_*)}}(1+o(1)).$$

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- The proofs rely on probabilistic estimates by Petrov (1975).

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The result is obtained from the concentration of a random vector X uniformly distributed on $B_M^n(nR)$ on a thin shell of radius $\sqrt{n\mathrm{Var} Z}$.

Thin-shell concentration

If K is an isotropic convex body and X is uniformly distributed on K then $\|X\|_2$ concentrates around $\sqrt{n}L_K$.

- Klartag (2006)

$$\mathbb{P}\left(\left|\frac{\|X\|_2}{\sqrt{n}L_K} - 1\right| \geq t\right) \leq Ce^{-ct^{3.33}n^{0.33}}, \quad \forall 0 < t < 1$$

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If X is uniformly distributed on B_p^n then

- Naor (2007). If $p \geq 2$

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  and if $1 \leq p < 2$

$$\mathbb{P}\left(\left|\frac{\|X\|_2^2}{\mathbb{E}\|X\|_2^2} - 1\right| \geq t\right) \leq Ce^{-c\psi(n,t)}, \quad \forall t > 0,$$

  where

$$\psi(n,t) = \begin{cases} n^{3-p}t^2 & \text{if } 0 < t \leq n^{-\frac{(6-p)(2-p)}{2(4-p)}} \\ n^{\left(2-\frac{p}{2}\right)\frac{p}{2}} t^{\frac{p}{2}} & \text{if } n^{-\frac{(6-p)(2-p)}{2(4-p)}} < t \leq n^{-\left(1-\frac{p}{2}\right)} \\ nt & \text{if } t > n^{-\left(1-\frac{p}{2}\right)} \end{cases}$$

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Theorem (Alonso, Prochno, 2021)

Let M be an Orlicz function such that $M(x) \in \Omega(x^2)$ as $x \to \infty$, $R > 0$, and X_n uniformly distributed on $B_M^n(nR)$. If $\frac{1}{\sqrt{n}} \ll t_n \ll 1$ then

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n\mathrm{Var}Z} - 1\right| \geq t_n\right) \leq |\alpha_*|\sqrt{2\pi n\varphi_M''(\alpha_*)}e^{-\frac{(\mathrm{Var}Z)^2 t_n^2 n}{2\mathrm{Var}[Z^2]}(1+o(1))}(1+o(1))$$

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$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n\mathrm{Var}Z} - 1\right| \geq t_n\right) \geq |\alpha_*|\sqrt{2\pi n \varphi_M''(\alpha_*)}e^{-(\mathrm{Var}Z)^2 t_n^2 n(-\alpha_* + o(1))}(1+o(1))$$

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The previous results include the case of $B_p^n(n) := B_{|\cdot|^p}(n) = n^{\frac{1}{p}} B_p^n$ for $p \geq 2$, being

$$\mathrm{Var} Z = \frac{p^{\frac{2}{p}} \Gamma\left(1+\frac{3}{p}\right)}{3\Gamma\left(1+\frac{1}{p}\right)}, \quad \mathrm{Var}[Z^2] = \frac{p^{\frac{4}{p}} \left(9\Gamma\left(1+\frac{5}{p}\right)\Gamma\left(1+\frac{1}{p}\right) - 5\Gamma\left(1+\frac{3}{p}\right)^2\right)}{45\Gamma\left(1+\frac{1}{p}\right)^2}, \quad \forall p \geq 1$$

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If $\frac{4}{3} < p < 2$ and $\frac{1}{n^{1/4}} \ll t_n \ll \frac{n^{\frac{3p-4}{4(4-p)}}}{n^{1/4}}$ then

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n \mathrm{Var} Z} - 1\right| \geq t_n\right) \geq \sqrt{\frac{2\pi n}{p}} e^{-\frac{(\mathrm{Var} Z)^2 t_n^2 n}{p}(1+o(1))}(1+o(1))$$

Proofs (Asymptotic value of $L_{B_M^n(nR)}$)

Take $(Z_i)_{i=1}^n$ independent copies of Z and the centered random variables

- $Y_i^{(1)} = M(Z_i) - R$
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$$\mathbb{E}\left[\chi_{(-\infty,0]}\left(\sum_{i=1}^n Y_i^{(1)} \right) e^{-\alpha_* \sum_{i=1}^n Y_i^{(1)}} \right] = \frac{1 + o(1)}{|\alpha_*| \sqrt{2\pi n \varphi_M''(\alpha_*)}}$$

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By Chebyshev's inequality

$$\mathbb{P}\left(\left(\sum_{i=1}^{n} Y_i^{(2)}\right)^2 \geq n^2 t^2\right) \leq \frac{n\mathrm{Var}[Y_1^{(2)}]}{n^2 t^2} = \frac{\mathrm{Var}[Z^2]}{nt^2}$$

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Thus, for every $t > 0$

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n} - \mathrm{Var} Z\right| \geq t\right) \leq \frac{|\alpha_*|\mathrm{Var}[Z^2]\sqrt{2\pi\varphi_M''(\alpha_*)}}{t^2\sqrt{n}}(1 + o(1)).$$

Proofs (Asymptotic value of $L_{B_M^n(nR)}$)

In particular,

$$\mathbb{P}\left(\frac{\|X_n\|_2}{\sqrt{n}} \geq t\right) \to \begin{cases} 1 & \text{if } t \in [0, \sqrt{\mathrm{Var} Z}) \\ 0 & \text{if } t \in (\sqrt{\mathrm{Var} Z}, \infty) \end{cases}$$

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Using the fact that $L_{B_M^n(nR)} \leq C$ in order to apply the dominated convergence theorem

$$\mathbb{E}\left[\frac{\|X_n\|_2^2}{n}\right] = \int_0^\infty 2t\mathbb{P}\left(\frac{\|X_n\|_2}{\sqrt{n}} \geq t\right) dt \to \int_0^{\sqrt{\mathrm{Var} Z}} 2t dt = \mathrm{Var} Z.$$

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Since $\frac{B_M^n(nR)}{|B_M^n(nR)|^{\frac{1}{n}}}$ is isotropic and $|B_M^n(nR)|^{\frac{1}{n}} \to e^{\varphi_M(\alpha_*)-\alpha_* R} = \frac{\int_{\mathbb{R}} e^{\alpha^* M(x)} dx}{e^{\alpha_* R}}$,

$$L_{B_M^n(nR)} = \frac{1}{|B_M^n(nR)|^{\frac{1}{n}}} \left(\mathbb{E}\left[\frac{\|X_n\|_2^2}{n}\right]\right)^{\frac{1}{2}} \to \frac{e^{\alpha_* R}}{\int_{\mathbb{R}} e^{\alpha^* M(x)} dx} \sqrt{\mathrm{Var}(Z)}$$

Large Deviation Principles

- $(X_n)_{n=1}^{\infty}$ a sequence of random vectors in $\mathbb{R}^d$
- $\mathcal{I} : \mathbb{R}^d \to [0, \infty]$ lower semi-continuous with compact level sets $\{x \in \mathbb{R}^d \ : \ \mathcal{I}(x) \leq \alpha\}$
- $s : \mathbb{N} \to [0, \infty)$

Definition

$(X_n)_{n=1}^{\infty}$ satisfies a Large Deviation Principle (LDP) with speed $s(n)$ and rate function $\mathcal{I}$ if for every A measurable

$$
\begin{aligned}
-\inf_{x \in A^\circ} \mathcal{I}(x) \quad &\leq \quad \liminf_{n \to \infty} \frac{\log \mathbb{P}(X_n \in A)}{s(n)} \\
&\leq \quad \limsup_{n \to \infty} \frac{\log \mathbb{P}(X_n \in A)}{s(n)} \leq -\inf_{x \in \overline{A}} \mathcal{I}(x).
\end{aligned}
$$

Large Deviation Principles

Theorem (Cramér)

Let $(Y_n)_{n=1}^{\infty}$ be a sequence of independent copies of a centered random variable Y such that

$$\Lambda(u) = \log \mathbb{E}[e^{uY}] < \infty$$

on a neighborhood of 0. Then, for every $t > 0$

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| \geq t\right)}{n} = -\inf_{|s|\geq t} \Lambda^*(s),$$

where Λ^* is the Legendre transform of Λ.

Large Deviation Principles

Theorem (Petrov)

Let $(Y_n)_{n=1}^{\infty}$ be a sequence of independent copies of a centered random vector in $\mathbb{R}^d$ Y with invertible covariance matrix C such that

$$\Lambda(u) = \log \mathbb{E}[e^{\langle u, Y \rangle}] < \infty$$

on a neighborhood of 0. Then, if $1 \ll s_n \ll \sqrt{n}$, the sequence of random vectors

$$\frac{1}{s_n \sqrt{n}} \sum_{i=1}^{n} Y_i$$

satisfies an LDP with speed s_n^2 and rate function $\mathcal{I}(x) = \frac{1}{2}\langle x, C^{-1}x \rangle$.

Large Deviation Principles

Given a speed $s(n)$, two sequences $(X_n)_{n=1}^{\infty}$, $(Y_n)_{n=1}^{\infty}$ of random vectors in $\mathbb{R}^d$ are called exponentially equivalent if

$$\limsup_{n\to\infty} \frac{1}{s(n)} \log(\mathbb{P}(\|X_n - Y_n\|_2 > \delta)) = -\infty$$

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Lemma

Let X_n and Y_n be two sequence of $\mathbb{R}^d$-valued random vectors and assume that X_n satisfies an LDP with speed $s(n)$ and rate function $\mathcal{I}_X$ and that X_n and Y_n are exponentially equivalent. Then Y_n satisfies an LDP with the same speed and the same rate function.

Proofs (Upper bound)

If $M \in \Omega(x^2)$ as $x \to \infty$ then, since $Y_i^{(2)} = Z_i^2 - \mathrm{Var} Z$

$$\mathbb{E}[e^{uY_1^{(2)}}] = e^{-u\mathrm{Var} Z}\mathbb{E}[e^{uZ^2}] = \frac{e^{-u\mathrm{Var} Z} \int_{\mathbb{R}} e^{ux^2} e^{\alpha_* M(x)} dx}{\int_{\mathbb{R}} e^{\alpha_* M(x)} dx} < \infty$$

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Thus, if $\frac{1}{\sqrt{n}} \ll t_n \ll 1$, taking $1 \ll s_n = t_n\sqrt{n} \ll \sqrt{n}$, by Petrov's theorem $\left(\frac{1}{s_n\sqrt{n}} \sum_{i=1}^n Y_i^{(2)}\right)_{n=1}^{\infty}$ satisfies an LDP with speed s_n^2 and rate function

$$\mathcal{I}(x) = \frac{x^2}{2\mathrm{Var}[Y_i^{(2)}]} = \frac{x^2}{2\mathrm{Var}[Z^2]}.$$

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Then

$$\frac{1}{s_n^2} \log \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n Y_i^{(2)}\right| \geq t_n\right)$$

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Then

$$\frac{1}{s_n^2} \log \mathbb{P}\left(\left|\frac{1}{s_n\sqrt{n}} \sum_{i=1}^n Y_i^{(2)}\right| \geq 1\right) \to -\mathcal{I}(1) = -\frac{1}{2\mathrm{Var}[Z^2]}.$$

Proofs (Upper bound)

Equivalently,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i^{(2)}\right| \geq t_n\right) = e^{-\frac{s_n^2}{2\mathrm{Var}[Z^2]}(1+o(1))} = e^{-\frac{t_n^2 n}{2\mathrm{Var}[Z^2]}(1+o(1))}.$$

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In conclusion,

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n} - \mathrm{Var}Z\right| \geq t_n\right) \leq |\alpha_*|\sqrt{2\pi n\varphi_M''(\alpha_*)}e^{-\frac{t_n^2 n}{2\mathrm{Var}[Z^2]}(1+o(1))}(1+o(1))$$

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Proofs (Upper bound)

Remark: If $M \in \Omega(x^4)$ it is possible pass from

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n\mathrm{Var}Z} - 1\right| \geq t_n\right) \leq |\alpha_*|\sqrt{2\pi n\varphi_M''(\alpha_*)}e^{-\frac{(\mathrm{Var}Z)^2 t_n^2 n}{2\mathrm{Var}[Z^2]}(1+o(1))}(1+o(1))$$

to

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{\mathbb{E}\|X_n\|_2^2} - 1\right| \geq t_n\right) \leq |\alpha_*|\sqrt{2\pi n\varphi_M''(\alpha_*)}e^{-\frac{(\mathrm{Var}Z)^2 t_n^2 n}{64A^2}(1+o(1))}(1+o(1))$$

for any sequence $(t_n)_{n=1}^{\infty}$, being $A = \inf\left\{\lambda > 0 \,:\, \mathbb{E}\exp\left(\frac{Y_1^{(2)}}{\lambda}\right)^2 \leq 2\right\}$.

Proofs (Lower bound)

For every $t > 0$

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n} - \mathrm{Var} Z\right| \geq t\right) \leq \frac{\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n Y_i^{(2)}\right| \geq t\right)}{\mathbb{E}\left[\chi_{(-\infty,0]}\left(\sum_{i=1}^n Y_i^{(1)}\right) e^{-\alpha_* \sum_{i=1}^n Y_i^{(1)}}\right]}$$

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For every $t > 0$ and every $(r_n)_{n=1}^\infty$

$$\begin{aligned} &\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n} - \mathrm{Var} Z\right| \geq t\right) \\ \geq & e^{\alpha_* r_n} \frac{\mathbb{P}\left(-1 \leq \frac{1}{r_n}\sum_{i=1}^n Y_i^{(1)} \leq 0, \left|\frac{1}{n}\sum_{i=1}^n Y_i^{(2)}\right| \geq t\right)}{\mathbb{E}\left[\chi_{(-\infty,0]}\left(\sum_{i=1}^n Y_i^{(1)}\right) e^{-\alpha_* \sum_{i=1}^n Y_i^{(1)}}\right]} \end{aligned}$$

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$$\mathbb{E}\left[\chi_{(-\infty,0]}\left(\sum_{i=1}^n Y_i^{(1)}\right) e^{-\alpha_* \sum_{i=1}^n Y_i^{(1)}}\right] = \frac{1 + o(1)}{|\alpha_*|\sqrt{2\pi n \varphi_M''(\alpha_*)}}$$

Proofs (Lower bound)

Calling $s_n = t_n\sqrt{n}$ and $v_n = \frac{r_n}{\sqrt{n}}$ we have

$$\mathbb{P}\left(-1 \leq \frac{1}{r_n}\sum_{i=1}^{n} Y_i^{(1)} \leq 0, \left|\frac{1}{n}\sum_{i=1}^{n} Y_i^{(2)}\right| \geq t_n\right) =$$

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for any $0 < \varepsilon < 1$, where

$$A_\varepsilon = \{(x, y) \in \mathbb{R}^2 \,:\, x \in [-1, -\varepsilon], \, |y| \geq 1\}.$$

Proofs (Lower bound)

- By Petrov's theorem, if $\frac{1}{\sqrt{n}} \ll \frac{r_n}{n} \ll 1$,

$$\left(\frac{1}{v_n \sqrt{n}} \sum_{i=1}^{n} Y_i^{(1)}, 0 \right)_{n=1}^{\infty}$$

satisfies an LDP with speed v_n^2 and rate function

$$\mathcal{I}(x, y) = \begin{cases} \frac{x^2}{2\mathrm{Var}[M(Z)^2]} & \text{if } y = 0 \\ \infty & \text{if } y \neq 0 \end{cases}$$

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- If $\frac{r_n}{n} \ll t_n$ then

$$\left(\frac{1}{v_n \sqrt{n}} \sum_{i=1}^{n} Y_i^{(1)}, 0 \right)_{n=1}^{\infty} \quad \text{and} \quad \left(\frac{1}{v_n \sqrt{n}} \sum_{i=1}^{n} Y_i^{(1)}, \frac{1}{s_n \sqrt{n}} \sum_{i=1}^{n} Y_i^{(2)} \right)_{n=1}^{\infty}$$

  are exponentially equivalent with speed v_n^2.

Proofs (Lower bound)

Then,

$$\mathbb{P}\left(\left(\frac{1}{v_n\sqrt{n}}\sum_{i=1}^{n} Y_i^{(1)}, \frac{1}{s_n\sqrt{n}}\sum_{i=1}^{n} Y_i^{(2)}\right) \in A_\varepsilon\right) \geq e^{-\left(\frac{\varepsilon^2}{2\mathrm{Var}[M(Z)^2]}+o(1)\right)v_n^2}$$

Proofs (Lower bound)

Then,

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In conclusion, if $\frac{1}{\sqrt{n}} \ll \frac{r_n}{n} \ll t_n \ll 1$,

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n} - \mathrm{Var} Z\right| \geq t_n\right)$$

$$\geq |\alpha_*|\sqrt{2\pi n \varphi_M''(\alpha_*)} e^{\alpha_* r_n} e^{-\left(\frac{\varepsilon^2}{2\mathrm{Var}[M(Z)^2]}+o(1)\right)\frac{r_n}{n} r_n}(1+o(1))$$

Proofs (Lower bound)

Then,

$$\mathbb{P}\left(\left(\frac{1}{v_n\sqrt{n}}\sum_{i=1}^{n} Y_i^{(1)}, \frac{1}{s_n\sqrt{n}}\sum_{i=1}^{n} Y_i^{(2)}\right) \in A_\varepsilon\right) \geq e^{-\left(\frac{\varepsilon^2}{2\mathrm{Var}[M(Z)^2]}+o(1)\right)\frac{r_n}{n}r_n}$$

In conclusion, if $\frac{1}{\sqrt{n}} \ll \frac{r_n}{n} \ll t_n \ll 1$,

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n} - \mathrm{Var} Z\right| \geq t_n\right)$$

$$\geq |\alpha_*|\sqrt{2\pi n\varphi_M''(\alpha_*)}e^{\alpha_* r_n(1+o(1))}(1+o(1))$$

Proofs (Lower bound)

Taking $r_n = t_n^2 n$ with $\frac{1}{n^{1/4}} \ll t_n \ll 1$ we obtain

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Equivalently,

$$\mathbb{P}\left(\left|\frac{\|X_n\|_2^2}{n\mathrm{Var}\,Z} - 1\right| \geq t_n\right) \geq |\alpha_*|\sqrt{2\pi n \varphi_M''(\alpha_*)} e^{-(\mathrm{Var}\,Z)^2 t_n^2 n(-\alpha_*+o(1))}(1+o(1)).$$

Proofs ($B_p^n(n)$, $1 \leq p < 2$)

If $M(t) = |t|^p$ with $1 \leq p < 2$ we use the following LDP

Theorem (Eichelsbacher-Löwe)

Let $(Y_n)_{n=1}^{\infty}$ be a sequence of independent copies of a centered random variable Y with positive variance and let $(s_n)_{n=1}^{\infty}$ be such that $1 \ll s_n \ll \sqrt{n}$. Assume that

$$\lim_{n\to\infty} \log\left(n\mathbb{P}\left(|Y| \geq s_n\sqrt{n}\right)\right) = -\infty$$

Then, the sequence of random variables

$$\frac{1}{s_n\sqrt{n}}\sum_{i=1}^{n} Y_i$$

satisfies an LDP with speed s_n^2 and rate function $\mathcal{I}(x) = \frac{x^2}{2\mathrm{Var}\, Y}$.

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The condition $\frac{1}{\sqrt{n}} \ll t_n \ll \frac{n^{\frac{p}{2(4-p)}}}{\sqrt{n}}$ ensures that we can use the latter LDP.

THANKS FOR YOUR ATTENTION!