A unified treatment for L_p Brunn-Minkowski type inequalities

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Outline of this talk

This talk is based on the joint work with Du Zou, published in Comm. Anal. Geom. 26 (2018), 435-460.

- ⋆ **The classical Brunn-Minkowski inequality**
- ⋆ L_p **Brunn-Minkowski type inequalities**
- ⋆ **Our work: The** L_p **transference principle**
- ⋆ **Applications**

1. The classical Brunn-Minkowski inequality

1. The classical Brunn-Minkowski inequality

1.1 The Brunn-Minkowski theorem

1. The Brunn-Minkowski theorem

Suppose that K, L are convex bodies (compact convex sets with non-empty interiors) in $\mathbb{R}^n$ and $0 < \alpha < 1$. Then

$$V_n((1-\alpha)K + \alpha L)^{\frac{1}{n}} \geq (1-\alpha)V_n(K)^{\frac{1}{n}} + \alpha V_n(L)^{\frac{1}{n}},$$

with equality if and only if K, L are homothetic.

- The theorem was discovered in 1887 (for $n \leq 3$) by H. Brunn (German, 1862 - 1939. Brunnian link in knot theory).
- Minkowski gave an analytical proof for the n-dimensional case and characterized the equality condition in 1896.

1.1 The Brunn-Minkowski theorem

- The theorem states that: $K \mapsto V_n(K)^{\frac{1}{n}}$ is concave.
- K, L are homothetic, if $L = \lambda K + x$, where $\lambda > 0,\ x \in \mathbb{R}^n$.
- V_n is the Lebesgue measure on $\mathbb{R}^n$.
- Minkowski combination: $\beta K + \gamma L = \{\beta x + \gamma y : x \in K,\ y \in L\}$.
- Intuitive explanation:

$$A(C(o,l)+B(o,r)) = A(C)+4lr+A(B) \geq A(C)+2\sqrt{\pi}lr+A(B).$$

$$= A(C)+2\sqrt{A(C)A(B)}+A(B)$$

1.1 The Brunn-Minkowski theorem

- The equivalent forms of the Brunn-Minkowski inequality:

$$V_n(K+L)^{\frac{1}{n}} \geq V_n(K)^{\frac{1}{n}} + V_n(L)^{\frac{1}{n}}. \tag{1}$$

$$V_n(\beta K+\gamma L)^{\frac{1}{n}} \geq \beta V_n(K)^{\frac{1}{n}} + \gamma V_n(L)^{\frac{1}{n}}. \tag{2}$$

$$V_n((1-\alpha)K+\alpha L)^{\frac{1}{n}} \geq (1-\alpha) V_n(K)^{\frac{1}{n}} + \alpha V_n(L)^{\frac{1}{n}}. \tag{3}$$

$$V_n((1-\alpha)K+\alpha L) \geq V_n(K)^{1-\alpha} V_n(L)^{\alpha}. \tag{4}$$

$$V_n((1-\alpha)K+\alpha L) \geq \min\{V_n(K),\ V_n(L)\}. \tag{5}$$

$$V_n((1-\alpha)K+\alpha L) \geq 1, \qquad \text{if } V_n(K)=V_n(L)=1. \tag{6}$$

1.2 The significance of the Brunn-Minkowski theorem

1. To get a full impression of the impact of the Brunn-Minkowski inequality in geometry and analysis, read the survey article

• R. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002), 355-405. [cited 358]

2. The Brunn-Minkowski inequality is a powerful tool for conquering problems involving metric quantities, such as volume, surface area and mean width.

1.2 The significance of the Brunn-Minkowski theorem

3. It quickly implies the classical isoperimetric inequality.

The isoperimetric inequality

Suppose that K is a convex body in $\mathbb{R}^n$ and B is the unit ball. Then

$$\left(\frac{S(K)}{S(B)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(K)}{V_n(B)}\right)^{\frac{1}{n}},$$

with equality if and only if K is a ball in $\mathbb{R}^n$.

1.2 The significance of the Brunn-Minkowski theorem

- $S(K)$ is the surface area of K, defined by

$$S(K) = \lim_{\epsilon \to 0^+} \frac{V_n(K + \epsilon B) - V_n(K)}{\epsilon}.$$

- R. Osserman (American, 1926 – 2011), The isoperimetric inequality, Bull. Amer. Math. Soc. 84 (1978), 1182-1238.

1.2 The significance of the Brunn-Minkowski theorem

4. The isoperimetric inequality for compact domains with C^1 boundaries is equivalent to the Sobolev inequality.

The Sobolev inequality

If f is a C^1 function on $\mathbb{R}^n$ with compact support, then

$$\left(\int_{\mathbb{R}^n} |\nabla f(x)|dx\right)^n \geq n^n \omega_n \parallel f \parallel_{\frac{n}{n-1}} .$$

• Sobolev (Soviet, 1908 – 1989) inequalities, relating norms in Sobolev spaces, are used to prove Sobolev embedding theorem.

1.2 The significance of the Brunn-Minkowski theorem

5. It quickly implies the Brunn concavity theorem.

The Brunn concavity theorem

Suppose that K is a convex bodies in $\mathbb{R}^n$. Then for $\forall u \in \mathbb{S}^{n-1}$,

$$F_K(t) = V_{n-1}(K \cap (tu + u^{\perp}))^{\frac{1}{n-1}}, \quad t \in \mathbb{R},$$

is concave on its support.

- H. Brunn, About ovals and eggforms, München, 1887.

1.2 The significance of the Brunn-Minkowski theorem

6. It is equivalent to the Minkowski first inequality.

The Minkowski first inequality

Suppose that K, L are convex bodies in $\mathbb{R}^n$. Then

$$V_1(K,L) \geq V_n(K)^{\frac{n-1}{n}} V_n(L)^{\frac{1}{n}},$$

with equality if and only if K, L are homothetic.

- $V_1(K,L)$ is the first mixed volume of K and L, defined by

$$V_1(K,L) = \frac{1}{n} \lim_{\epsilon \to 0^+} \frac{V_n(K+\epsilon L) - V_n(K)}{\epsilon}.$$

1.2 The significance of the Brunn-Minkowski theorem

7. An extension of the Brunn-Minkowski theorem

The general Brunn-Minkowski theorem

Suppose that E, F are Lebesgue measurable sets in $\mathbb{R}^n$ and their sum $E + F$ is also measurable. Then

$$V_n(E+F)^{\frac{1}{n}} \geq V_n(E)^{\frac{1}{n}} + V_n(F)^{\frac{1}{n}}.$$

• The inequality was first proved in 1935 by Lyusternik (Soviet, 1899-1981).

• Hadwiger and Ohmann found a simple and beautiful proof in 1956, now called the Hadwiger-Ohmann cut.

1.2 The significance of the Brunn-Minkowski theorem

8. Further extensions of the Brunn-Minkowski theorem

- C. Borell, The Brunn-Minkowski inequality in Gauss space. Invent. Math. 30 (1975), 207 – 216.
- B. Berndtsson, A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. Invent. Math. 200 (2015), 149 – 200.
- A. Figalli, D. Jerison, Quantitative stability for the Brunn-Minkowski inequality. Adv. Math. 314 (2017), 1-47.
- A. Figalli, Quantitative stability results for the Brunn-Minkowski inequality. Proceedings of ICM, 2014.

1.2 The significance of the Brunn-Minkowski theorem

• A. Okounkov, Brunn-Minkowski inequality for multiplicities. Invent. Math. 125 (1996), 405 – 411.

• Y. Ollivier, C. Villani, A curved Brunn-Minkowski inequality on the discrete hypercube. SIAM J. Discrete Math. 26 (2012), 983 – 996.

• A. Colesanti, P. Salani, The Brunn-Minkowski inequality for p-capacity of convex bodies. Math. Ann. 327 (2003), 459 – 479.

• L. Caffarelli, D. Jerison, E. Lieb, On the case of equality in the Brunn-Minkowski inequality for capacity. Adv. Math. 117 (1996), 193 – 207.

1.3 Proof of the Brunn-Minkowski theorem

1. **Proof. (By induction)**

• Due to H. Kneser (German, 1898 – 1973) and W. Süss (German, 1895 – 1958, the first director of the MRI of Oberwolfach) in 1932.

Step 1. The volume of convex body K in $\mathbb{R}^n$ can be expressed as

$$V_n(K) = \int_{\xi_K}^{\eta_K} V_{n-1}(K \cap H_{u,t})dt,$$

where $u \in \mathbb{S}^{n-1}$, $H_{u,t} = \{x \in \mathbb{R}^n : x \cdot u = t\}$, $\xi_K, \eta_K \in \mathbb{R}$.

1.3 Proof of the Brunn-Minkowski theorem

Step 2. For convex bodies K, L in $\mathbb{R}^n$ and $0 < \alpha < 1$, there holds the inclusion relation

$$((1-\alpha)K + \alpha L) \cap H_{u,t} \supseteq (1-\alpha)(K \cap H_{u,t}) + \alpha(L \cap H_{u,t}).$$

Step 3. According to the monotonicity of volume and the induction hypothesis, the inequality can be derived.

• H. Kneser, W. Süss, Die Volumina in linearen Scharen konvexer Körper, Mat. Tidsskr. B (1932), 19-25.

2. The L_p Brunn-Minkowski type inequalities

2. The L_p Brunn-Minkowski type inequalities

2.1 The L_p combination

1. In 1962, Firey introduced the L_p combinations of convex bodies, and established the L_p Brunn-Minkowski inequality.

• W. Firey, p-means of convex bodies, Math. Scand. 10 (1962), 17-24.

• For $1 \leq p < \infty$, the L_p combination $\alpha \cdot_p K +_p \beta \cdot_p L$ of convex bodies K, L, is defined by $\alpha \cdot_p K = \alpha^{1/p} K, \quad \beta \cdot_p L = \beta^{1/p} L,$

$$K +_p L = \{(1-\gamma)^{\frac{p-1}{p}} x + \gamma^{\frac{p-1}{p}} y \ : \ x \in K, \ y \in L, \ 0 \leq \gamma \leq 1\}.$$

• $\alpha \cdot_p K +_p \beta \cdot_p L$ is a convex body with origin in its interior.

2.1 The L_p combination

2. Further developments of the L_p Brunn-Minkowski theory were greatly impelled by E. Lutwak in 1990s.

• E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, JDG 38 (1993), 131-150. [cited 416]

• E. Lutwak, The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas, Adv. Math. 118 (1996), 244-294. [cited 343]

2.2 The L_p Brunn-Minkowski theorem

The L_p Brunn-Minkowski theorem

Suppose that $K, L \in \mathcal{K}_o^n$, $p \in (1, +\infty)$ and $\alpha \in (0,1)$, then

$$V_n((1-\alpha)\cdot_p K +_p \alpha \cdot_p L)^{\frac{p}{n}} \geq (1-\alpha)V_n(K)^{\frac{p}{n}} + \alpha V_n(L)^{\frac{p}{n}},$$

with equality if and only if K, L are dilates.

- $\mathcal{K}_o^n$: the set of convex bodies with the origin in their interiors.
- The functional $V_n^{\frac{p}{n}} : \mathcal{K}_o^n \to [0, \infty)$ is concave.
- An equivalent form: $V_n(K +_p L)^{\frac{p}{n}} \geq V_n(K)^{\frac{p}{n}} + V_n(L)^{\frac{p}{n}}$.

2.3 The L_p Brunn-Minkowski type inequalities

1. For a convex body $K \in \mathcal{K}^n$, its quermassintegrals $W_0(K)$, $W_1(K)$, ..., $W_{n-1}(K)$ are defined by

$$W_{n-j}(K) = \frac{\omega_n}{\omega_j} \int_{\mathrm{G}_{n,j}} V_j(K|\xi) d\mu_j(\xi)$$

for $j = 1, \ldots, n-1$. $W_0(K) = V_n(K)$ and $W_n(K) = V_n(B)$.

• The Grassmann manifold $\mathrm{G}_{n,j}$ is endowed with the normalized Haar measure μ_j; $K|\xi$ is the orthogonal projection of K onto ξ.

• If K has a $\mathcal{C}^2$ boundary, $W_i(K)$ are the integrals of elementary symmetric functions of the principal curvatures over ∂K.

2.3 The L_p Brunn-Minkowski type inequalities

2. The Brunn-Minkowski inequality for quermassintegrals

Suppose that $K, L \in \mathcal{K}^n$, and $j = 1, \ldots, n-1$. Then

$$W_{n-j}(K+L)^{\frac{1}{j}} \geq W_{n-j}(K)^{\frac{1}{j}} + W_{n-j}(L)^{\frac{1}{j}},$$

with equality if and only if K and L are homothetic.

- The functional ${W_{n-j}}^{\frac{1}{j}}$ is concave on $\mathcal{K}^n$.
- R. Schneider, Convex bodies: the Brunn-Minkowski theory, Cambridge University Press, 2014.

2.3 The L_p Brunn-Minkowski type inequalities

3. The L_p Brunn-Minkowski inequality for quermassintegrals

Suppose that $K, L \in \mathcal{K}_o^n$, $j = 1, \ldots, n-1$ and $1 < p < \infty$. Then

$$W_{n-j}(K +_p L)^{\frac{p}{j}} \geq W_{n-j}(K)^{\frac{p}{j}} + W_{n-j}(L)^{\frac{p}{j}},$$

with equality if and only if K and L are dilates.

- The functional ${W_{n-j}}^{\frac{p}{j}}$ is concave on $\mathcal{K}_o^n$.

- E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, JDG 38 (1993), 131-150.

3. Our work: The L_p transference principle

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3.1 Motivation: A unified treatment for L_p BM type inequalities

Known Facts:

The functionals $V_n^{\frac{1}{n}}$, $W_{n-j}^{\frac{1}{j}}, \cdots$, are concave on $\mathcal{K}^n$ w.r.t. Minkowski combinations, then $V_n^{\frac{p}{n}}$, $W_{n-j}^{\frac{p}{j}}, \cdots$, are concave on $\mathcal{K}_o^n$ w.r.t. L_p combinations.

Question

Suppose a functional F is concave on $\mathcal{K}^n$, w.r.t. Minkowski combinations. Under what conditions F^p is concave on $\mathcal{K}_o^n$, w.r.t. L_p combinations? ...... If answered, then a unified treatment is obtained!

3.2 The L_p transference principle

1. Given a functional $F : \mathcal{K}^n \to [0, \infty)$, we say that F is

(1) positively homogeneous, provided

$$F(\alpha K) = \alpha F(K), \quad \alpha > 0.$$

(2) increasing, provided

$$K \subseteq L \quad \Longrightarrow \quad F(K) \leq F(L).$$

Moreover, F is strictly increasing, provided

$$K \subsetneq L \quad \Longrightarrow \quad F(K) < F(L).$$

3.2 The L_p transference principle

(3) *p*-concave, provided

$$F((1-\alpha)\cdot_p K +_p \alpha \cdot_p L)^p \geq (1-\alpha)F(K)^p + \alpha F(L)^p, \quad \alpha \in (0,1).$$

As usual, 1-concave is called concavity for brevity.

(4) translation invariant, provided

$$F(K+x) = F(K), \quad x \in \mathbb{R}^n.$$

3.2 The L_p transference principle

2. Theorem 3.1 (L_p transference principle)

Suppose that $F : \mathcal{K}^n \to [0, \infty)$ is positively homogeneous, increasing and concave, and $p \in (1, \infty)$. If $K, L \in \mathcal{K}_o^n$, then

$$F((1-\alpha)\cdot_p K +_p \alpha \cdot_p L)^p \geq (1-\alpha)F(K)^p + \alpha F(L)^p, \quad \alpha \in (0,1).$$

Furthermore, if $F : \mathcal{K}_o^n \to [0, \infty)$ is strictly increasing, the equality holds if and only if K and L are dilates.

- Equivalent form: $F(K +_p L)^p \geq F(K)^p + F(L)^p$.

3.2 The L_p transference principle

Lemma 3.1

Suppose that $K, L \in \mathcal{K}_o^n$ and $1 \le p < \infty$. Then, F is p-concave, if and only if $F_{p;K,L}$ is concave.

- $F_{p;K,L} : [0,1] \to [0,\infty)$, $F_{p;K,L}(\alpha) = F\left((1-\alpha) \cdot_p K +_p \alpha \cdot_p L\right)^p$.

Lemma 3.2

Suppose that $K, L \in \mathcal{K}_o^n$, $1 < p < \infty$ and $0 < \alpha < 1$. Then

$$(1-\alpha) \cdot_p K +_p \alpha \cdot_p L \supseteq (1-\alpha)K + \alpha L,$$

with equality if and only if $K = L$.

3.3 Characterizations of equality conditions

Remark: For many L_p Brunn-Minkowski type inequalities, equality only occurs when the convex bodies are dilates, not homothetic. This phenomenon can be completely characterized.

Theorem 3.2

Suppose that $F : \mathcal{K}^n \to [0, \infty)$ is positively homogeneous, increasing and concave, and $p \in (1, \infty)$. Then the following assertions are equivalent.

(1) For $K, L \in \mathcal{K}_o^n$, the function $F_{p;K,L}$ is affine if and only if K and L are dilates.

(2) When restricted to $\mathcal{K}_o^n$, the functional F is strictly increasing.

3.3 Characterizations of equality conditions

Theorem 3.3

Suppose $F : \mathcal{K}^n \to [0, \infty)$ is translation invariant, positively homogeneous, increasing and concave, and $p \in (1, \infty)$. Then the following assertion (1) implies assertion (2).

(1) For $K, L \in \mathcal{K}^n$, the function $F_{1;K,L}$ is affine if and only if K and L are homothetic.

(2) For $K, L \in \mathcal{K}^n_o$, the function $F_{p;K,L}$ is affine if and only if K and L are dilates.

4. Applications

4. Applications

4.1 An application to harmonic quermassintegrals

1. For a convex body $K \in \mathcal{K}^n$, Hadwiger introduced the harmonic quermassintegrals $\hat{W}_0$, $\hat{W}_1$, ..., $\hat{W}_{n-1}$, defined by $\hat{W}_0(K) = V_n(K)$, and for $j = 1, \ldots, n-1$,

$$\hat{W}_{n-j}(K) = \frac{\omega_n}{\omega_j} \left(\int_{\mathrm{G}_{n,j}} V_j(K|\xi)^{-1} d\mu_j(\xi) \right)^{-1}.$$

• H. Hadwiger (Swiss, 1908-1981), Vorlesungen über Inhalt, Oberäche, und Isoperimetrie, Springer-Verlag, Berlin, 1957.

4.1 An application to harmonic quermassintegrals

2. The Brunn-Minkowski inequality for harmonic quermassintegrals

Suppose that $K, L \in \mathcal{K}^n$ and $j \in \{1, \ldots, n-1\}$. Then

$$\hat{W}_j(K+L)^{\frac{1}{n-j}} \geq \hat{W}_j(K)^{\frac{1}{n-j}} + \hat{W}_j(L)^{\frac{1}{n-j}},$$

with equality if and only if K and L are homothetic.

- Let $F(K) = \hat{W}_j(K)^{\frac{1}{n-j}}$. Then F is positively homogeneous, strictly increasing and concave on $\mathcal{K}^n$.

4.1 An application to harmonic quermassintegrals

From **Theorem 3.1** and **Theorem 3.2**, it follows that

3. Theorem 4.1

Suppose that $K, L \in \mathcal{K}_o^n$, $j \in \{1, \ldots, n-1\}$ and $1 < p < \infty$. Then

$$\hat{W}_j(K +_p L)^{\frac{p}{n-j}} \geq \hat{W}_j(K)^{\frac{p}{n-j}} + \hat{W}_j(L)^{\frac{p}{n-j}},$$

with equality if and only if K and L are dilates.

4.2 An application to affine quermassintegrals

1. For a convex body $K \in \mathcal{K}^n$, Lutwak introduced the affine quermassintegrals $\Phi_0, \ldots, \Phi_{n-1}$, defined by $\Phi_0(K) = V_n(K)$, and for $j = 1, \ldots, n-1$,

$$\Phi_{n-j}(K) = \frac{\omega_n}{\omega_j} \left(\int_{\mathrm{G}_{n,j}} V_j(K|\xi)^{-n} d\mu_j(\xi) \right)^{-\frac{1}{n}}.$$

- Note that all the $\Phi_j(K)$ are affine invariant, i.e.,

$$\Phi_j(TK) = \Phi_j(K), \quad \text{for all } T \in \mathrm{SL}(n).$$

4.2 An application to affine quermassintegrals

2. The Brunn-Minkowski inequality for affine quermassintegrals

Suppose that $K, L \in \mathcal{K}^n$ and $j \in \{1, \ldots, n-1\}$. Then

$$\Phi_j(K+L)^{\frac{1}{n-j}} \geq \Phi_j(K)^{\frac{1}{n-j}} + \Phi_j(L)^{\frac{1}{n-j}}.$$

If $j = n-1$, equality holds in each inequality if and only if $w_K = \lambda w_L$ for some constant $\lambda > 0$. If $1 \leq j < n-1$, equality holds in each inequality if and only if K and L are homothetic.

• E. Lutwak, A general isepiphanic inequality, Proc. Amer. Math. Soc. 90 (1984), 415-421.

4.2 An application to affine quermassintegrals

• For $K \in \mathcal{K}^n$, let $F(K) = \Phi_j(K)^{\frac{1}{n-j}}$. Then functional F is positively homogeneous, strictly increasing and concave on $\mathcal{K}^n$.

From **Theorem 3.1** and **Theorem 3.2**, it follows that

Theorem 4.2

Suppose that $K, L \in \mathcal{K}_o^n$, $j \in \{1, \ldots, n-1\}$ and $1 < p < \infty$. Then

$$\Phi_j(K +_p L)^{\frac{p}{n-j}} \geq \Phi_j(K)^{\frac{p}{n-j}} + \Phi_j(L)^{\frac{p}{n-j}},$$

with equality if and only if K and L are dilates.

4.3 An application to moments of inertia

1. From classic mechanics, we know that for each convex body K in $\mathbb{R}^n$, its moment of inertia, $I(K)$, is defined by

$$I(K) = \int_K |x - c_K|^2 dx,$$

where c_K denotes the centroid of K.

2.The Brunn-Minkowski inequality for moments of inertia

Suppose $K, L \in \mathcal{K}^n$. Then $I(K+L)^{\frac{1}{n+2}} \geq I(K)^{\frac{1}{n+2}} + I(L)^{\frac{1}{n+2}}$.

• H. Hadwiger, Konkave Eikörperfunktionale und höhere Trägheitsmomente, Comment. Math. Helv. 30 (1956), 285-296.

4.3 An application to moments of inertia

• For $K \in \mathcal{K}^n$, let $F(K) = I(K)^{\frac{1}{n+2}}$. Then F is positively homogeneous and concave.

• If K is origin-symmetric, then the centroid c_K of K is at the origin. when the domain of F is restricted to the the class of origin-symmetric convex bodies $\mathcal{K}^n_{os}$, then $F : \mathcal{K}^n_{os} \to (0, \infty)$ is strictly increasing.

4.3 An application to moments of inertia

From **Theorem 3.1** and **Theorem 3.2**, it follows that

Theorem 4.3

Suppose that $K, L \in \mathcal{K}_o^n$ are origin-symmetric and $1 < p < \infty$. Then

$$I(K +_p L)^{\frac{p}{n+2}} \geq I(K)^{\frac{p}{n+2}} + I(L)^{\frac{p}{n+2}},$$

with equality if and only if K and L are dilates.

4.4 An application to capacities

1. The *q-capacity* of a convex body K in $\mathbb{R}^n$, for $1 \leq q < n$, is defined by

$$\mathrm{Cap}_q(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^q dx \right\},$$

where the infimum is taken over all nonnegative functions f such that $f \in L^{\frac{nq}{n-q}}(\mathbb{R}^n)$, $\nabla f \in L^q(\mathbb{R}^n; \mathbb{R}^n)$ and $K \subseteq \{x : f(x) \geq 1\}$.

2. The Brunn-Minkowski type inequality for capacities

Suppose $K, L \in \mathcal{K}^n$, and $1 \leq q < n$. Then

$$\mathrm{Cap}_q(K+L)^{\frac{1}{n-q}} \geq \mathrm{Cap}_q(K)^{\frac{1}{n-q}} + \mathrm{Cap}_q(L)^{\frac{1}{n-q}},$$

with equality if and only if K and L are homothetic.

4.4 An application to capacities

- C. Borell, Capacitary inequalities of the Brunn-Minkowski type, Math. Ann. 263 (1983), 179-184.
- L. Caffarelli, D. Jerison, E. Lieb, On the case of equality in the Brunn-Minkowski inequality for capacity, Adv. Math. 117 (1996), 193-207.
- A. Colesanti, P. Salani, The Brunn-Minkowski inequality for p-capacity of convex bodies, Math. Ann. 327 (2003), 459-479.
- L. Evans, R. Gariepy, Measure theory and fine properties of functions, CRC Press, Boca Raton, 1992.

4.4 An application to capacities

- For $K \in \mathcal{K}^n$, let $F(K) = \mathrm{Cap}_q(K)^{\frac{1}{n-q}}$.
- By Evans and Gariepy, F is positively homogeneous, increasing, concave and translation invariant.

From **Theorem 3.1** and **Theorem 3.3**, it follows that

Theorem 4.4

Suppose $K, L \in \mathcal{K}_o^n$, $1 \le q < n$, and $1 < p < \infty$. Then

$$\mathrm{Cap}_q(K +_p L)^{\frac{p}{n-q}} \ge \mathrm{Cap}_q(K)^{\frac{p}{n-q}} + \mathrm{Cap}_q(L)^{\frac{p}{n-q}},$$

with equality if and only if K and L are dilates.

Acknowledgement

Thank You !!!