Inequalities for the Radon transform on convex sets.

Alexander Koldobsky

University of Missouri-Columbia

joint work with A.Giannopoulos and A.Zvavitch

The main results

Theorem 1. Let K and L be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and L, respectively, so that $\|g\|_\infty = g(0) = 1$. Then

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}.$$

Here $|K|$ is volume of proper dimension, Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $\mathcal{BP}_k^n$ is the class of generalized k-intersection bodies in $\mathbb{R}^n$.

Theorem 1. Let K and L be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and L, respectively, so that $\|g\|_\infty = g(0) = 1$. Then

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}.$$

Here $|K|$ is volume of proper dimension, Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $\mathcal{BP}_k^n$ is the class of generalized k-intersection bodies in $\mathbb{R}^n$.

Theorem 2. Let K and L be two bounded Borel sets in $\mathbb{R}^n$. Let f and g be two bounded non-negative measurable functions on K and L, respectively, and assume that $\|g\|_1 > 0$ and $\|g\|_\infty = 1$. For every $1 \le k \le n-1$ we have that

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \left(C \cdot {\rm ovr}(K)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g},$$

where $C > 0$ is an absolute constant.

If K is symmetric convex then both constants are $\le \sqrt{n}$.

Constants

The outer volume ratio distance from a star body K to the class $\mathcal{BP}_k^n$ is defined by

$$d_{\text{ovr}}(K,\mathcal{BP}_k^n) = \inf\left\{ \left(\frac{|D|}{|K|}\right)^{1/n} :\ K \subset D,\ D \in \mathcal{BP}_k^n \right\}.$$

Constants

The outer volume ratio distance from a star body K to the class $\mathcal{BP}_k^n$ is defined by

$$d_{\rm ovr}(K,\mathcal{BP}_k^n) = \inf\left\{ \left(\frac{|D|}{|K|}\right)^{1/n} :\ K\subset D,\ D\in \mathcal{BP}_k^n \right\}.$$

It was proved in K., Paouris, Zymonopoulou (2013) that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$d_{\rm ovr}(K,\mathcal{BP}_k^n) \le C\sqrt{\frac{n}{k}}\log^{\frac{3}{2}}\left(\frac{en}{k}\right).$$

The outer volume ratio distance from a star body K to the class $\mathcal{BP}_k^n$ is defined by

$$d_{\rm ovr}(K,\mathcal{BP}_k^n)=\inf\left\{\left(\frac{|D|}{|K|}\right)^{1/n}:\ K\subset D,\ D\in\mathcal{BP}_k^n\right\}.$$

It was proved in K., Paouris, Zymonopoulou (2013) that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$d_{\rm ovr}(K,\mathcal{BP}_k^n)\le C\sqrt{\frac{n}{k}}\log^{\frac{3}{2}}\left(\frac{en}{k}\right).$$

For many classes of bodies, this distance is bounded by an absolute constant. It was proved in K. (2015) that for unconditional convex bodies K one has $d_{\rm ovr}(K,\mathcal{BP}_k^n)\le e$. If K is the unit ball of an n-dimensional subspace of $L_p,\ p>2$, the distance is less than $c\sqrt{p}$, where $c>0$ is an absolute constant, as shown by E.Milman and K.-Pajor. The unit balls of subspaces of L_p with $0<p\le 2$ belong to the classes $\mathcal{BP}_k^n$ for all k,n (K., 1998), so the distance for these bodies is equal to 1.

The outer volume ratio distance from a star body K to the class $\mathcal{BP}_k^n$ is defined by

$$d_{\mathrm{ovr}}(K,\mathcal{BP}_k^n) = \inf\left\{\left(\frac{|D|}{|K|}\right)^{1/n} :\ K \subset D,\ D \in \mathcal{BP}_k^n\right\}.$$

It was proved in K., Paouris, Zymonopoulou (2013) that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$d_{\mathrm{ovr}}(K,\mathcal{BP}_k^n) \le C\sqrt{\frac{n}{k}}\log^{\frac{3}{2}}\left(\frac{en}{k}\right).$$

For many classes of bodies, this distance is bounded by an absolute constant. It was proved in K. (2015) that for unconditional convex bodies K one has $d_{\mathrm{ovr}}(K,\mathcal{BP}_k^n) \le e$. If K is the unit ball of an n-dimensional subspace of $L_p,\ p > 2$, the distance is less than $c\sqrt{p}$, where $c > 0$ is an absolute constant, as shown by E.Milman and K.-Pajor. The unit balls of subspaces of L_p with $0 < p \le 2$ belong to the classes $\mathcal{BP}_k^n$ for all k,n (K., 1998), so the distance for these bodies is equal to 1.

The outer volume ratio of K is defined by

$$\mathrm{ovr}(K) = \inf\left\{\left(\frac{|\mathcal{E}|}{|K|}\right)^{1/n} :\ \mathcal{E} \text{ origin} - \text{symmetric ellipsoids such that } K \subseteq \mathcal{E}\right\}.$$

Slicing problem: Does there exist an absolute constant C so that for any $n \in \mathbb{N}$, any $0 < k < n$ and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-k}{n}} \le C^k \max_{H \in Gr n-k} |K \cap H| \ ?$$

Slicing inequality for arbitrary functions

Slicing problem: Does there exist an absolute constant C so that for any $n \in \mathbb{N}$, any $0 < k < n$ and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-k}{n}} \leq C^k \max_{H \in Gr n-k} |K \cap H| \ ?$$

Bourgain (1991) proved that $C \leq O(n^{1/4} \log n)$. Klartag (2006) removed the logarithmic term from Bourgain's estimate. Chen (2021) proved that $C \leq o(n^{\epsilon})$ for every $\epsilon > 0$, as the dimension goes to infinity.

Slicing inequality for arbitrary functions

Slicing problem: Does there exist an absolute constant C so that for any $n \in \mathbb{N}$, any $0 < k < n$ and any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-k}{n}} \leq C^k \max_{H \in Grn-k} |K \cap H| \ ?$$

Bourgain (1991) proved that $C \leq O(n^{1/4} \log n)$. Klartag (2006) removed the logarithmic term from Bourgain's estimate. Chen (2021) proved that $C \leq o(n^{\epsilon})$ for every $\epsilon > 0$, as the dimension goes to infinity.

An extension of the slicing problem to arbitrary functions was proved in K. (2015): for any $n \in \mathbb{N}$, any star body K in $\mathbb{R}^n$ and any non-negative continuous function f on K,

$$\int_K f \ \leq \ \left(e \cdot d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)\right)^k |K|^{k/n} \max_{H \in Grn-k} \int_{K \cap H} f.$$

If K is symmetric convex, by John's theorem,

$$\int_K f \ \leq \ (e\sqrt{n})^k |K|^{k/n} \max_{H \in Grn-k} \int_{K \cap H} f.$$

Slicing inequality from Theorem 1

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}.$$

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}.$$

Slicing inequality: Put $L = B_2^n$ and $g \equiv 1$:

$$\int_K f \le \frac{n}{n-k} \frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|} \left(d_{\rm ovr}(K, BP_k^n)\right)^k |K|^{\frac{k}{n}} \max_H \int_{K\cap H} f.$$

Note that the constant $\frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|}$ is less than 1, and $\frac{n}{n-k} \le e^k$.

Slicing inequality from Theorem 1

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \le \frac{n}{n-k} \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)\right)^k \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L \cap H} g}.$$

Slicing inequality: Put $L = B_2^n$ and $g \equiv 1$:

$$\int_K f \le \frac{n}{n-k} \frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|} \left(d_{\mathrm{ovr}}(K, BP_k^n)\right)^k |K|^{\frac{k}{n}} \max_H \int_{K\cap H} f.$$

Note that the constant $\frac{|B_2^n|^{\frac{n-k}{n}}}{|B_2^{n-k}|}$ is less than 1, and $\frac{n}{n-k} \le e^k$.

Mean value inequality for the Radon transform: Let $K = L$, and $g \equiv 1$. Then

$$\frac{\int_K f}{|K|} \le \frac{n}{n-k} \left(d_{\mathrm{ovr}}(K, BP_k^n)\right)^k \max_H \frac{\int_{K\cap H} f}{|K \cap H|}.$$

The Busemann-Petty problem (1956): Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and suppose that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller than the corresponding one for L, i.e. $|K\cap\xi^\perp|\le|L\cap\xi^\perp|$ for every $\xi\in S^{n-1}$. Here $\xi^\perp=\{x\in\mathbb{R}^n:\ \langle x,\xi\rangle=0\}$ is the central hyperplane perpendicular to $\xi\in S^{n-1}$. Does it necessarily follow that the n-dimensional volume of K is smaller than the volume of L, i.e. $|K|\le|L|$?

The Busemann-Petty problem (1956): Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and suppose that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller than the corresponding one for L, i.e. $|K \cap \xi^\perp| \leq |L \cap \xi^\perp|$ for every $\xi \in S^{n-1}$. Here $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$. Does it necessarily follow that the n-dimensional volume of K is smaller than the volume of L, i.e. $|K| \leq |L|$?

The answer is affirmative if the dimension $n \leq 4$, and it is negative when $n \geq 5$. Ball, Bourgain, Gardner, Giannopoulos, K., Larman, Lutwak, Papadimitrakis, Rogers, Schlumprecht, Zhang

The Busemann-Petty problem (1956): Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and suppose that the $(n-1)$-dimensional volume of every central hyperplane section of K is smaller than the corresponding one for L, i.e. $|K \cap \xi^\perp| \le |L \cap \xi^\perp|$ for every $\xi \in S^{n-1}$. Here $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$. Does it necessarily follow that the n-dimensional volume of K is smaller than the volume of L, i.e. $|K| \le |L|$?

The answer is affirmative if the dimension $n \le 4$, and it is negative when $n \ge 5$. Ball, Bourgain, Gardner, Giannopoulos, K., Larman, Lutwak, Papadimitrakis, Rogers, Schlumprecht, Zhang

An extension of the Busemann-Petty problem to arbitrary functions was found by Zvavitch (2005). Suppose that f is an even continuous strictly positive function on $\mathbb{R}^n$, and K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ so that

$$\int_{K \cap \xi^\perp} f \le \int_{L \cap \xi^\perp} f, \qquad \forall \xi \in S^{n-1}.$$

Does it necessarily follow that $\int_K f \le \int_L f$? The answer is the same as for the volume, affirmative if $n \le 4$ and negative if $n \ge 5$.

Isomorphic Busemann-Petty problem: Does there exist an absolute constant C so that for any dimension n, any $0 < k < n$, and any pair of origin-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$|K \cap H| \le |L \cap H|, \qquad \forall H \in Gr_{n-k},$$

we have

$$|K|^{\frac{n-k}{n}} \le C^k |L|^{\frac{n-k}{n}} ?$$

Isomorphic Busemann-Petty problem: Does there exist an absolute constant C so that for any dimension n, any $0 < k < n$, and any pair of origin-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$|K \cap H| \leq |L \cap H|, \qquad \forall H \in Gr_{n-k},$$

we have

$$|K|^{\frac{n-k}{n}} \leq C^k |L|^{\frac{n-k}{n}}?$$

K. (2015):

$$|K|^{\frac{n-k}{n}} \leq \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k |L|^{\frac{n-k}{n}}.$$

Isomorphic Busemann-Petty problem: Does there exist an absolute constant C so that for any dimension n, any $0 < k < n$, and any pair of origin-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$|K \cap H| \leq |L \cap H|, \qquad \forall H \in Gr_{n-k},$$

we have

$$|K|^{\frac{n-k}{n}} \leq C^k |L|^{\frac{n-k}{n}} ?$$

K. (2015):

$$|K|^{\frac{n-k}{n}} \leq \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \right)^k |L|^{\frac{n-k}{n}}.$$

We show a little stronger result:

$$\left(\frac{|K|}{|L|} \right)^{\frac{n-k}{n}} \leq \left(d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \right)^k \max_{H \in Gr_{n-k}} \frac{|K \cap H|}{|L \cap H|}.$$

K., Zvavitch (2015): Suppose that f is a non-negative continuous function on $\mathbb{R}^n$, $0 < k < n$, K, L are star bodies in $\mathbb{R}^n$ so that

$$\int_{K\cap H} f \le \int_{L\cap H} f, \qquad \forall H \in Gr_{n-k}.$$

Then

$$\int_K f \le \left(d_{BM}(K, \mathcal{BP}_k^n)\right)^k \int_L f,$$

where

$$d_{BM}(K, \mathcal{BP}_k^n) = \inf\{a > 0 : \exists D \in \mathcal{BP}_k^n : D \subset K \subset aD\}$$

is the Banach-Mazur distance from K to the class of generalized k-intersection bodies.

Another version

Another version of the isomorphic Busemann-Petty problem was proved in K., Paouris, Zvavitch (2019+). Let K and L be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and L, respectively, so that $\|g\|_\infty = g(0) = 1$,

$$\int_{K\cap H} f \le \int_{L\cap H} g, \qquad \forall H \in Gr_{n-k},$$

then

$$\int_K f \le \frac{n}{n-k} \left(d_{\rm ovr}(K, BP_k^n)\right)^k |K|^{\frac{k}{n}} \left(\int_L g\right)^{\frac{n-k}{n}}.$$

Another version of the isomorphic Busemann-Petty problem was proved in K., Paouris, Zvavitch (2019+). Let K and L be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and L, respectively, so that $\|g\|_\infty = g(0) = 1$,

$$\int_{K\cap H} f \leq \int_{L\cap H} g, \qquad \forall H \in Gr_{n-k},$$

then

$$\int_K f \leq \frac{n}{n-k} \left(d_{\rm ovr}(K, BP_k^n)\right)^k |K|^{\frac{k}{n}} \left(\int_L g\right)^{\frac{n-k}{n}}.$$

Follows from Theorem 1:

$$\frac{\int_K f}{\left(\int_L g\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \leq \frac{n}{n-k} \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \max_{H\in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}.$$

Definitions

We need several definitions and facts. A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$$

is a continuous function on $\mathbb{R}^n$. We use the polar formula for the volume $|K|$ of a star body K :

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta.$$

If f is a continuous function on K, then

$$\int_K f = \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) dr \right) d\theta.$$

For $1 \le k \le n-1$, the $(n-k)$-**dimensional spherical Radon transform** $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1}\cap H} g(x)\ dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

Intersection bodies

For $1 \le k \le n-1$, the $(n-k)$-**dimensional spherical Radon transform** $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1}\cap H} g(x)\ dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

For an integrable function f and any $H \in Gr_{n-k}$,

$$\int_{K\cap H} f = R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot)\ dr\right)(H).$$

For $1 \le k \le n-1$, the $(n-k)$-**dimensional spherical Radon transform** $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1}\cap H} g(x)\ dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

For an integrable function f and any $H \in Gr_{n-k}$,

$$\int_{K\cap H} f = R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot)\ dr\right)(H).$$

The class of intersection bodies was introduced by Lutwak, and generalized by Zhang, as follows. An origin symmetric star body D in $\mathbb{R}^n$ is called a **generalized** k**-intersection body**, and we write $D \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure ν_D on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_D^{-k} g(x)\ dx = \int_{Gr_{n-k}} R_{n-k}g(H)\ d\nu_D(H).$$

When $k = 1$ we get the original Lutwak's class of **intersection bodies** $\mathcal{BP}_1^n = \mathcal{I}_n$

Proof of Theorem 1.

For a small $\delta > 0$, let $D \in \mathcal{BP}_k^n$ be a body such that $K \subset D$ and

$$|D|^{\frac{1}{n}} \le (1+\delta)\ d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)\ |K|^{\frac{1}{n}}, \tag{1}$$

and let ν_D be the measure on Gr_{n-k} corresponding to D by the definition of intersection bodies. Let $\varepsilon = \max_H \int_{K\cap H} f / \int_{L\cap H} g$, then

$$\int_{K\cap H} f \le \varepsilon \int_{L\cap H} g, \qquad \forall H \in Gr_{n-k}.$$

Writing this in terms of the Radon transform

$$R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot)\ dr\right)(H) \le \varepsilon\, R_{n-k}\left(\int_0^{\|\cdot\|_L^{-1}} r^{n-k-1} g(r\ \cdot)\ dr\right)(H)$$

for every $H \in Gr_{n-k}$. Integrating both sides of the latter inequality with respect to ν_D and using the definition of intersection bodies, we get

$$\int_{S^{n-1}} \|x\|_D^{-k} \left(\int_0^{\|x\|_K^{-1}} r^{n-k-1} f(rx)\ dr\right) dx \tag{2}$$

$$\le \varepsilon \int_{S^{n-1}} \|x\|_D^{-k} \left(\int_0^{\|x\|_L^{-1}} r^{n-k-1} g(rx)\ dr\right) dx,$$

which is equivalent to

$$\int_K \|x\|_D^{-k} f(x)dx \le \varepsilon \int_L \|x\|_D^{-k} g(x)dx. \tag{3}$$

Since $K \subset D$, we have $1 \ge \|x\|_K \ge \|x\|_D$ for every $x \in K$. Therefore,

$$\int_K \|x\|_D^{-k} f(x)dx \ge \int_K \|x\|_K^{-k} f(x)dx \ge \int_K f.$$

On the other hand, by a result of V.Milman-Pajor (recall that $g(0) = \|g\|_\infty = 1$),

$$\left(\frac{\int_L \|x\|_D^{-k} g(x)dx}{\int_D \|x\|_D^{-k} dx} \right)^{1/(n-k)} \le \left(\frac{\int_L g(x)dx}{\int_D dx} \right)^{1/n}.$$

Since $\int_D \|x\|_D^{-k} dx = \frac{n}{n-k}|D|$, we can estimate the right-hand side of (3) by

$$\int_L \|x\|_D^{-k} g(x)dx \le \varepsilon \frac{n}{n-k} \left(\int_L g \right)^{\frac{n-k}{n}} |D|^{\frac{k}{n}}.$$

Applying (1) and sending δ to zero, we see that the latter inequality in conjunction with (3) implies

$$\int_K f \le \varepsilon\ \frac{n}{n-k} \left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k |K|^{\frac{k}{n}}.$$

Now recall that $\varepsilon = \max_{H \in Gr_{n-k}} \frac{\int_{K\cap H} f}{\int_{L\cap H} g}$.

□

We say that a compact set K with volume 1 in $\mathbb{R}^n$ is in isotropic position if for each $\xi \in S^{n-1}$

$$\int_K \langle x, \xi \rangle^2 dx = L_K^2$$

where L_K is a constant that is called the isotropic constant of K.

We say that a compact set K with volume 1 in $\mathbb{R}^n$ is in isotropic position if for each $\xi \in S^{n-1}$

$$\int_K \langle x, \xi \rangle^2 dx = L_K^2$$

where L_K is a constant that is called the isotropic constant of K.

Hensley has proved that there exist absolute constants $c_1, c_2 > 0$ so that for any origin-symmetric convex body K in $\mathbb{R}^n$ in isotropic position and any $\xi \in S^{n-1}$

$$\frac{c_1}{L_K} \le |K \cap \xi^\perp| \le \frac{c_2}{L_K}.$$

The following inequality was proved by E. Milman.

Theorem

For any origin-symmetric isotropic convex body K in $\mathbb{R}^n$

$$L_K \le C\ d_{\mathrm{ovr}}(K, \mathcal{I}_n),$$

where C is an absolute constant.

Proof : Using

$$\left(\frac{|K|}{|D|}\right)^{\frac{n-k}{n}} \le \left(d_{\rm ovr}(K,\mathcal{BP}_k^n)\right)^k \max_{H\in Gr_{n-k}} \frac{|K\cap H|}{|D\cap H|}.$$

with $k=1$ and Hensley's theorem, for any origin-symmetric isotropic convex bodies K,D in $\mathbb{R}^n$

$$1 \le d_{\rm ovr}(K,\mathcal{I}_n) \max_{\xi\in S^{n-1}} \frac{|K\cap\xi^\perp|}{|D\cap\xi^\perp|} \le d_{\rm ovr}(K,\mathcal{I}_n)\frac{\frac{c_2}{L_K}}{\frac{c_1}{L_D}},$$

where $c_1,c_2>0$ are absolute constants, so

$$\frac{L_K}{L_D} \le C\ d_{\rm ovr}(K,\mathcal{I}_n).$$

Now put $D=B_2^n/|B_2^n|^{\frac{1}{n}}$, and use the fact that L_D is bounded by an absolute constant.

□